UNIT-I

PART A

1. **Form a partial differential equation by eliminating the arbitrary constants from** $(x-a)^2+(y-b)^2=z^2\cot^2\alpha$.

Solution:

$(x-a)^2+(y-b)^2=z^2\cot^2\alpha \rightarrow (1)$

$(x-a)=zp\cot^2\alpha \rightarrow (2)$

$(y-b)=zq\cot^2\alpha \rightarrow (3)$

$(1)\Rightarrow (zp\cot^2\alpha)^2+(zq\cot^2\alpha)^2=z^2\cot^2\alpha$

$z^2p^2\cot^4\alpha+z^2q^2\cot^4\alpha=z^2\cot^2\alpha$

$\cot^2\alpha(p^2+q^2)=1$

$(p^2+q^2)=\tan^2\alpha$

2. **Form a partial differential equation by eliminating the arbitrary constants from** $z=(x^2+a)(y^2+b)$

Solution:

Given that $z=(x^2+a)(y^2+b)$

$z=(x^2+a)(y^2+b)$

$p=\frac{\partial z}{\partial x}=(2x)(y^2+b)\rightarrow (1)$

$q=\frac{\partial z}{\partial y}=(x^2+a)(2y)\rightarrow (2)$

$(1)\Rightarrow (y^2+a)=\frac{p}{2x}$

$(2)\Rightarrow (x^2+a)=\frac{q}{2y}$

$z=\left(\frac{p}{2x}\right)\left(\frac{q}{2y}\right)=\frac{pq}{4xy}$

$4xyz=pq$.

3. **Form a partial differential equation by eliminating the arbitrary constants from** $z=ax^2+by^2$

Solution:.

Given $z=ax^2+by^2$ ...(1)

Differentiate (1) partially with respect to x, we get

$$\frac{\partial z}{\partial x}=a(2x)$$

$$\frac{p}{2x}=a$$

Differentiate (1) partially with respect to y, we get

$$\frac{\partial z}{\partial y}=b(2y)$$

$$\frac{q}{2y}=b$$

Substituting a & b in equation (1) we get

$$z=\frac{p}{2x}x^2+\frac{q}{2y}y^2$$

$$z=\frac{px+qy}{2}\Rightarrow 2z=px+qy$$

4. Form the partial differential equation by eliminating the arbitrary function f from $z=f\left(\frac{y}{x}\right)$.

Solution:

Given $z = f\left(\frac{y}{x}\right)$ ...(1)

Differentiate (1) partially with respect to x, we get

$$p = \frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$p = f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) \quad ...(2)$$

Differentiate (1) partially with respect to y, we get

$$q = \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$q = f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \quad ...(3)$$

$$\frac{(2)}{(3)} \Rightarrow \qquad \frac{p}{q} = \frac{f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)}{f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)}$$

$$\frac{p}{q} = \frac{-y}{x} \Rightarrow px + qy = 0$$

5. Find the particular integral of $(D^2-2DD'+D'^2)z=e^{x-y}$

Solution:

Given $(D^2 + 2DD' + D'^2)z = e^{x-y}$

$$P.I = \frac{1}{D^2 + 2DD' + D'^2} e^{x-y}$$

$$P.I = \frac{1}{1+2(1)(-1)+1} e^{x-y}$$

Replace $D = 1, D' = -1$

$$P.I = \frac{1}{0} e^{x-y}$$

$$= \frac{x}{2D + 2D' + 0} e^{x-y}$$

$$= \frac{x}{2-2} e^{x-y} \qquad \text{Replace } D = 1, D' = -1$$

$$= \frac{x}{0} e^{x-y}$$

$$= \frac{x^2}{2+0} e^{x-y}$$

$$P.I = \frac{x^2}{2} e^{x-y}$$

6. Form the partial differential equation by eliminating the arbitrary function f from $(z^2-xy)=f\left(\frac{x}{z}\right)$.

Solution:

Given $\varphi\left(z^2 - xy, \frac{x}{z}\right) = 0$

Let $u = z^2 - xy, v = \frac{x}{z}$

$$\frac{\partial u}{\partial x} = 2z\frac{\partial z}{\partial x} - y = 2zp - y$$

$$\frac{\partial u}{\partial y} = 2z\frac{\partial z}{\partial y} - x = 2zq - x$$

$$\frac{\partial v}{\partial x} = \frac{z(1) - x\frac{\partial z}{\partial x}}{z^2} = \frac{z - px}{z^2}$$

$$\frac{\partial v}{\partial y} = -\frac{x}{z^2}\frac{\partial z}{\partial x} = \frac{-xq}{z^2}$$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\begin{vmatrix} 2zp - y & \frac{z - px}{z^2} \\ 2zq - x & \frac{-xq}{z^2} \end{vmatrix} = 0$$

$$(2zp - y)\left(\frac{-xq}{z^2}\right) - (2zq - x)\left(\frac{z - px}{z^2}\right) = 0$$

$$-\frac{2xpq}{z} + \frac{xyq}{z^2} - (2zq - x)\left(\frac{z - px}{z^2}\right) = 0$$

$$x^2p - (xy - 2z^2)q = xz$$

7. Find the partial differential equation of the family of spheres having their centres on the z-axis.

Solution:

Let the centre of the sphere be $(0,0.c)$ a point on the $z - axis$ and r radius.

It's equation is $(x-0)^2 + (y-0)^2 + (z-c)^2 = r^2$

$$x^2 + y^2 + (z-c)^2 = r^2 \ldots\ldots\ldots\ldots\ldots(1)$$

Here c and r are constants.

Diff (1) p.w.r.to x we get

$$2x + 2(z-c)\frac{\partial z}{\partial x} = 0$$

$$x + (z-c)p = 0$$

$$(z-c)p = -x$$

$$(z-c) = -\frac{x}{p} \ldots\ldots\ldots\ldots\ldots(2)$$

Diff (1) p.w.r.to y we get

$$2y + 2(z-c)\frac{\partial z}{\partial y} = 0$$

$$y + (z-c)q = 0$$

$$(z-c)q = -y$$

$$(z-c) = -\frac{y}{q} \ldots\ldots\ldots\ldots\ldots(3)$$

Equating (2) & (3), we get

$$\therefore \frac{-x}{p} = \frac{-y}{q}$$

$$qx = py$$

8. What Solve $(D^2 - 7DD' + 6D'^2)z = 0$

Solution:

Let $D = m, D' = 1$

$m^2 - 7m + 6 = 0$

$(m-1)(m-6) = 0$

$m = 1,6$

Complementary function $z = f_1(y + 1x) + f_2(y + 6x)$

9. Solve $(D-1)(D-D'+1)z=0$.

Solution:

Given $(D-1)(D-D'+1)z = 0$

$\left(D-(0)D'-1\right)\left(D-(1)D'-(-1)\right)z = 0$

Here

$m_1 = 0, \quad m_2 = 1$

$c_1 = 1, \quad c_2 = -1$

$$z = e^{x} f_1(y+0x) + e^{-x} f_2(y+x)$$

10. Find the complete integral of $p+q=pq$.

Solution:

Let the complete solution be

$z = ax + by + c$..........(1)

Given $p + q = pq$

Let $p = a \,\&\, q = b$

$a + b = ab$

$b - ab = -a$

$b(1-a) = -a$

$b = \dfrac{-a}{1-a}$

$\Rightarrow b = \dfrac{a}{(a-1)}$

Substituting $b = \frac{a}{(a-1)}$ in equation (1) we get

$z = ax + \dfrac{a}{(a-1)}y + c$

11. Solve $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial z}{\partial x} = 0$

Solution:

Given Equation can be written as $(D^2 - DD' + D)Z = 0$

$D(D - D' + 1) = 0$

$(D - 0D' - 0)(D - (1)D' + 1) = 0$

Here $m_1 = 0, c_1 = 0 \,\&\, m_2 = 1, c_2 = -1$

$\therefore z = e^{c_1 x} f_1(y + m_1 x) + e^{c_2 x} f_2(y + m_2 x)$

$= e^{0x} f_1(y + 0x) + e^{-1x} f_2(y + 1x)$

$z = f_1(y) + e^{-x} f_2(y + x)$

12. Solve $(D - D')^3 z = 0$.

Solution:

The auxiliary equation is $(m-1)^3 = 0$.

$$m = 1,1,1$$

$\therefore z = \varphi_1(y + x) + x\varphi_2(y + x) + x^2\varphi_3(y + x)$

13. Find the complete solution of $p + q = 1$

Solution:

Let the complete solution be

$z = ax + by + c$...........(1)

Given $p + q = 1$

Let $p = a$ & $q = b$

$a + b = 1$

$b = 1 - a$

Substituting $b = 1 - a$ in equation (1) we get

$z = ax + (1 - a)y + c$

14. Solve $(D^3 - 2D^2 D')z = 0$

Solution:

Given $(D^3 - 2D^2 D')z = 0$

The auxiliary equation is $m^3 - 2m^2 = 0$

$m^3 - 2m^2 = 0$

$m^2(m - 2) = 0$

$m_1 = 0,\ m_2 = 0, m_3 = 2.$

$z = \phi_1(y) + x\phi_2(y) + \phi_3(y + 2x)$

15. Solve $(D^4 - D'^4)z = 0$.

Solution:

Let $D = m, D' = 1$

$m^4 - 1 = 0$

$(m^2 - 1)(m^2 + 1) = 0$

$m^2 - 1 = 0$ & $m^2 + 1 = 0$

$m = \pm 1, m = \pm i$

Complementary function $z = f_1(y + x) + f_2(y - x) + f_3(y + ix) + f_4(y - ix)$

PART B

16. Form a partial differential equation by eliminating the arbitrary function ϕ **from** $\phi(x^2 + y^2 + z^2, ax + by + cz) = 0$.

Solution:

Given $\varphi(x^2+y^2+z^2, ax+by+cz)=0$

Let $u=x^2+y^2+z^2, v=ax+by+cz$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}=0$$

$$\begin{vmatrix} 2x+2zp & a+cp \\ 2y+2zq & b+cq \end{vmatrix}=0$$

$$(2x+2zp)(b+cq)-(2y+2zq)(a+cp)=0$$

$$(2xb+2xcq+2zbp+2zcpq)-(2ay+2azq+2cyp+2czpq)=0$$

$$2xb+2xcq+2zbp+2zcpq-2ay-2azq-2cyp-2czpq=0$$

$$\div 2 \quad \Rightarrow xb+xcq+zbp-ay-azq-cyp=0$$

$$p(zb-cy)+q(xc-az)=ay-bx$$

17. Form a partial differential equation by eliminating the arbitrary function f & g from $z=x^2f(y)+y^2g(x)$.

Solution:

Given $z=x^2f(y)+y^2g(x)$ ....(1)

$$p=\frac{\partial z}{\partial x}=2xf(y)+y^2g'(x) \quad ...(2)$$

$$q=\frac{\partial z}{\partial y}=x^2f'(y)+2y\,g(x) \quad ...(3)$$

$$r=\frac{\partial^2 z}{\partial x^2}=2\,f(y)+\ y^2g''(x) \quad ...(4)$$

$$s=\frac{\partial^2 z}{\partial x\partial y}=2xf'(y)+2yg'(x) \quad ...(5)$$

$$t=\frac{\partial^2 z}{\partial y^2}=x^2f''(y)+2\,g(x) \quad ...(6)$$

$(2)x+(3)y \Rightarrow$

$$px+qy=2x^2f(y)+xy^2g'(x)+yx^2f'(y)+2y^2\,g(x)$$

$$px+qy=2[x^2f(y)+y^2\,g(x)]+xy[yg'(x)+xf'(y)]$$

$$px+qy=2z+xy\left(\frac{s}{2}\right)$$

$$2px+2qy=4z+xys$$

$$4z=2px+2qy-xys$$

18. Find the partial differential equation of all planes which are at a constant distance k from the origin.

Solution:

The equation of the plane which is at a constant distance k from the origin is

$$lx+my+nz=k$$

Where $l^2+m^2+n^2=1$

Let $l=cos\alpha=a, m=cos\beta=b, n=cos\gamma=c$

We get $ax + by + \sqrt{1-a^2-b^2}\ z = k$

$ie., ax + by + \sqrt{1-a^2-b^2}\ z = k$ ....(1)

Diff (1) w.r.t x we get

$$a + \sqrt{1-a^2-b^2}\ \frac{\partial z}{\partial x} = 0$$

$$\sqrt{1-a^2-b^2}\ p = -a$$

$$p = \frac{-a}{\sqrt{1-a^2-b^2}}$$

Diff (1) w.r.t y we get

$$b + \sqrt{1-a^2-b^2}\ \frac{\partial z}{\partial y} = 0$$

$$\sqrt{1-a^2-b^2}\ q = -a$$

$$q = \frac{-b}{\sqrt{1-a^2-b^2}}$$

$$\frac{a}{p} = \frac{b}{q} = \sqrt{1-a^2-b^2} = \lambda(say)$$

$$a = p\lambda, b = q\lambda, \sqrt{1-a^2-b^2} = \lambda$$

$$\sqrt{1-p^2\lambda^2-q^2\lambda^2} = \lambda$$

$$1-p^2\lambda^2-q^2\lambda^2 = \lambda^2$$

$$1-p^2\lambda^2-q^2\lambda^2-\lambda^2 = 0$$

$$1 = \lambda^2(p^2+q^2+1)$$

$$\lambda^2 = \frac{1}{(p^2+q^2+1)}$$

$$\lambda = \pm\frac{1}{\sqrt{(p^2+q^2+1)}}$$

$$\sqrt{p^2+q^2+1} = -\frac{1}{\lambda} \qquad \text{(here } \lambda \text{ is } -ve)$$

$$(1) \Rightarrow p\lambda x + q\lambda y - \lambda z = k$$

$$(px+qy-z)\lambda = k$$

$$(px+qy-z) = \frac{k}{\lambda}$$

$$z = px + qy + k\sqrt{p^2+q^2+1}$$

19. Find the singular integral of $z = px + qy + p^2 + q^2 + pq.$

Solution:

Given $z = px + qy + p^2 + \text{pq} + q^2$

This of the form $z = px + qy + f(p,q)$

Hence the complete integral is $z = ax + by + a^2 + \text{ab} + b^2$

Where a and b constants

Now $\quad \text{z} = \text{ax} + \text{by} + \text{a}^2 + \text{ab} + \text{b}^2$ ... ... (1)

Diff (1) w.r.t a we get

$0 = x + 2a + b$ ... ... (2)

Diff (1) w.r.t b we get

$0 = y + 2b + a \ldots\ldots\ldots\ldots\ldots\ldots.(3)$

Now $(3) \times 2 \Rightarrow \quad 2a + 4b + 2y = 0 \quad \ldots\ldots\ldots(4)$

Solving (2)& (4) we get

$(2) - (4) \Rightarrow \quad -3b + x - 2y = 0$

$3b = x - 2y$

$$b = \frac{x - 2y}{3}$$

$(3) \Rightarrow \quad 0 = y + 2\left(\frac{x-2y}{3}\right) + a$

$0 = \frac{3y+2x-4y}{3} + a$

$$a = -\left(\frac{2x - y}{3}\right)$$

$$a = \frac{y - 2x}{3}$$

Substituting in equation (1) we get

$$z = \left(\frac{y-2x}{3}\right)x + \left(\frac{x-2y}{3}\right)y + \left(\frac{y-2x}{3}\right)^2 + \left(\frac{y-2x}{3}\right)\left(\frac{x-2y}{3}\right) + \left(\frac{x-2y}{3}\right)^2$$

$$z = \frac{xy - 2x^2}{3} + \frac{xy - 2y^2}{3} + \frac{y^2 + 4x^2 - 4xy}{3} + \frac{xy - 2y^2 - 2x^2 + 4xy}{9} + \frac{x^2 + 4y^2 - 4xy}{9}$$

$=\frac{1}{9}[3xy - 6x^2 + 3xy - 6y^2 + y^2 + 4x^2 - 4xy + xy - 2y^2 - 2x^2 + 4xy + x^2 + 4y^2 - 4xy]$

$$= \frac{1}{9}[-3x^2 - 3y^2 + 3xy]$$

$$z = \frac{3}{9}[-x^2 - y^2 + xy]$$

$z = \frac{1}{3}[-x^2 - y^2 + xy]$

$3z = -x^2 - y^2 + xy$

$3z + x^2 + y^2 - xy = 0$

To find general integral:

Put $b = \varphi(a)$ in equation (1) we get

$z = ax + \varphi(a)y + a^2 + a\varphi(a) + [\varphi(a)]^2 \quad \ldots(5)$

Differentiating w.r.t a we get

$0 = x + \varphi'(a)y + 2a + a\varphi'(a) + \varphi(a) + 2\varphi(a)\varphi'(a) \quad \ldots(6)$

Eliminating a between(5) and (6) we get general solution

20. Find the singular solution to the equation $z = px + qy + \sqrt{1 + p^2 + q^2}$.

Solution:

Given $z = px + qy + \sqrt{1+p^2+q^2}$
This of the form $z = px + qy + f(p,q)$

Hence the complete integral is $z = ax + by + \sqrt{1+a^2+b^2}$

Where a and b constants

Now $z = ax + by + \sqrt{1+a^2+b^2} \ldots (1)$

Diff (1) w.r.t a we get

$$0 = x + 0 + \frac{1}{2}\frac{0+2a+0}{\sqrt{1+a^2+b^2}}$$

$$0 = x + \frac{a}{\sqrt{1+a^2+b^2}}$$

$$x = -\frac{a}{\sqrt{1+a^2+b^2}} \qquad \ldots (2)$$

Diff (1) w.r.t b we get

$$0 = 0 + y + \frac{1}{2}\frac{0+2b+0}{\sqrt{1+a^2+b^2}}$$

$$0 = y + \frac{b}{\sqrt{1+a^2+b^2}}$$

$$y = -\frac{b}{\sqrt{1+a^2+b^2}} \qquad \ldots (3)$$

$$x^2 + y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$1 - (x^2+y^2) = 1 - \frac{a^2+b^2}{1+a^2+b^2}$$

$$1 - x^2 - y^2 = \frac{1+a^2+b^2-a^2-b^2}{1+a^2+b^2}$$

$$1 - x^2 - y^2 = \frac{1}{1+a^2+b^2}$$

$$\sqrt{1-x^2-y^2} = \frac{1}{\sqrt{1+a^2+b^2}}$$

$$\sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}$$

Substituting in equation (2) we get

$$x = -a\sqrt{1-x^2-y^2}$$

$$a = -\frac{x}{\sqrt{1-x^2-y^2}}$$

Substituting in equation (3) we get

$$y = -b\sqrt{1-x^2-y^2}$$

$$b = -\frac{y}{\sqrt{1-x^2-y^2}}$$

Substituting in equation (1) we get

$$z = -\frac{x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$z = \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$$z = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2$$

$$x^2+y^2+z^2 = 1$$

To find general integral:

Put $b=\varphi(a)$in equation (1) we get

$$z = ax+\varphi(a)y+\sqrt{1+a^2+[\varphi(a)]^2} \qquad \ldots(4)$$

Differentiating w.r.t a we get

$$0 = x+\varphi'(a)y+\frac{1}{2}\frac{2a+2\varphi(a)\varphi'(a)}{\sqrt{1+a^2+[\varphi(a)]^2}} \qquad \ldots(5)$$

Eliminating a between (5) and (6) we get general solution.

21. Find the singular solution to the equation $z = px+qy+p^2q^2$

Solution:

Given $z = px+qy+p^2q^2$

This is of form $z = px+qy+f(p,q)$

Hence the complete integral is $z = ax+by+a^2b^2$

Where a and b are arbitrary constants

To find Singular solution

$$z = ax+by+a^2b^2 \qquad \ldots(1)$$

Differentiating with respect to 'a', we get

$$0 = x+2ab^2$$

$$x = -2ab^2 \qquad \ldots(2)$$

$$\frac{x}{b} = -2ab$$

Differentiating with respect to 'b', we get

$$0 = y+2a^2b$$

$$y = -2a^2b$$

$$\frac{y}{a} = -2ab \qquad \ldots(3)$$

$$\frac{x}{b} = \frac{y}{a} = -2ab = \frac{1}{k}(say)$$

$$a = ky; \quad b = kx$$

Put in equation (2) we get

$$x = -2k^3yx^2$$

$$k^3 = -\frac{1}{2xy}$$

Put a and b values in (1) we get

$z = kxy + kxy + k^4x^2y^2$

$z = 2kxy + kx^2y^2\left(-\frac{1}{2xy}\right)$

$z = 2kxy - \frac{k}{2}xy = \frac{3}{2}kxy$

$z^3 = \frac{27}{8}k^3x^3y^3 = \frac{27}{8}\left(-\frac{1}{2xy}\right)x^3y^3$

$z^3 = -\frac{27}{16}x^2y^2$

$16z^3 + 27x^2y^2 = 0$

This is the singular solution.

To get the general integral

Put $b = \varphi(a)$ in equation (1), we get

$z = ax + \varphi(a)y + a^2(\varphi(a))^2$ ...(4)

Differentiating w.r.to a we get

$0 = x + \varphi'(a)y + a^2 2\varphi(a)\varphi'(a) + c$ ...(5)

Eliminate a between (4) & (5) we get the general solution.

22. Solve the partial differential equation $x(y-z)p + y(z-x)q = z(x-y)$

Solution:

Given $x(y-z)p + y(z-x)q = z(x-y)$

The equation is of the form $Pp + Qq = R$

Where $P = x(y-z), Q = y(z-x), R = z(x-y)$

Lagrange's subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

Use Lagrange multipliers 1,1 ,1 we get

$$\frac{dx + dy + dz}{xy - xz + yz - xy + zx - yz} = \frac{dx + dy + dz}{0}$$

$\Rightarrow dx + dy + dz = 0$

Integrating we get

$\int dx + \int dy + \int dz = 0$

$x + y + z = c_1$

Use Lagrange multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ we get

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y - z + z - x + x - y} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\frac{1}{x}\,dx+\frac{1}{y}\,dy+\frac{1}{z}\,dz=0$$

Integrating we get

$$\int\frac{1}{x}\,dx+\int\frac{1}{y}dy+\int\frac{1}{z}dz=0$$

$$\log x+\log y+\log z=\log c_2$$

$$\log(\text{xyz})=\log c_2$$

$$\text{xyz}=c_2$$

Hence the general solution is $f(c_1,c_2)=0$

$$\varphi(x+y+z,xyz)=0$$

23. Solve the partial differential equation Solve the partial differential equation $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$

Solution:

Given $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$

The equation is of the form $Pp+Qq=R$

Where $P=x^2(y-z), Q=y^2(z-x),\ R=z^2(x-y)$

Lagrange's subsidiary equations are $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$$\frac{dx}{x^2(y-z)}=\frac{dy}{y^2(z-x)}=\frac{dz}{z^2(x-y)}$$

Use Lagrange multipliers $\frac{1}{x^2},\frac{1}{y^2},\frac{1}{z^2}$ we get

$$\frac{\frac{1}{x^2}\,dx+\frac{1}{y^2}\,dy+\frac{1}{z^2}\,dz}{(y-z)+(z-x)+(x-y)}=\frac{\frac{1}{x^2}\,dx+\frac{1}{y^2}\,dy+\frac{1}{z^2}\,dz}{0}$$

$$\frac{1}{x^2}\,dx+\frac{1}{y^2}\,dy+\frac{1}{z^2}\,dz=0$$

Integrating we get $\int\frac{1}{x^2}\,dx+\int\frac{1}{y^2}dy+\int\frac{1}{z^2}dz=0$

$$\int x^{-2}\,dx+\int y^{-2}\,dy+\int z^{-2}\,dz=0$$

$$\frac{x^{-2+1}}{-2+1}+\frac{y^{-2+1}}{-2+1}+\frac{z^{-2+1}}{-2+1}=0$$

$$-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}=-c_1$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=c_1$$

Use Lagrange multipliers $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ we get

$$\frac{\frac{1}{x}\,dx+\frac{1}{y}\,dy+\frac{1}{z}\,dz}{xy-xz+yz-yx+zx-zy}=\frac{\frac{1}{x}\,dx+\frac{1}{y}\,dy+\frac{1}{z}\,dz}{0}$$

$$\frac{1}{x}\,dx+\frac{1}{y}\,dy+\frac{1}{z}\,dz=0$$

Integrating we get

$$\int\frac{1}{x}\,dx+\int\frac{1}{y}dy+\int\frac{1}{z}dz=0$$

$$\log x+\log y+\log z=\log c_2$$

$$\log(\text{xyz})=\log c_2$$

$$\text{xyz}=c_2$$

Hence the general solution is $f(c_1,c_2)=0$

$$\varphi\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z},xyz\right)=0$$

24. Solve the partial differential equation Solve the partial differential equation $(x-2z)p+(2z-y)q=(y-x)$

Solution:

Given $(x-2z)p+(2z-y)q=y-x$

The equation is of the form $Pp+Qq=R$

Where $P=x-2z, Q=2z-y,\ R=y-x$

Lagrange's subsidiary equations are $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$$\frac{dx}{x-2z}=\frac{dy}{2z-y}=\frac{dz}{y-x}\quad \ldots\ldots\ldots\ldots..(1)$$

Use Lagrange multipliers 1,1,1we get

$$\frac{dx+\ dy+\ dz}{x-2z+2z-y+y-x}=\frac{dx+\ dy+\ dz}{0}$$

$$\Rightarrow dx+\ dy+\ dz=0$$

Integrating we get

$$\int dx+\int dy+\int dz=0$$

$$\Rightarrow\qquad x+y+z=c_1$$

Use Lagrange multipliers $y,x,2z$we get

$$\frac{ydx+\ xdy+2zdz}{y(x-2z)+x(2z-y)+2z(y-x)}=\frac{ydx+\ xdy+2zdz}{0}$$

$$ydx+\ xdy+2zdz=0$$

$$\int ydx+\int xdy+\int 2zdz=0$$

$$\Rightarrow\qquad xy+xy+2\frac{z^2}{2}=c_2$$

$$2xy+z^2=c_2$$

Hence the general solution is $\varphi(c_1,c_2)=0$

$$\varphi(x+y+z,2xy+z^2)=0$$

25. Solve the partial differential equation Solve the partial differential equation $(mz-ny)p+(nx-lz)q=ly-mx$

Solution:

Given $(mz-ny)p+(nx-lz)q=ly-mx$

The equation is of the form $Pp+Qq=R$

Where $P=mz-ny, Q=nx-lz,\ R=ly-mx$

Lagrange's subsidiary equations are $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

Use Lagrange multipliers x, y, z we get

$$\frac{x\,dx+y\,dy+z\,dz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}=\frac{x\,dx+y\,dy+z\,dz}{mxz-nxy+nxy-lyz+lyz-mxz}$$

$$=\frac{x\,dx+y\,dy+z\,dz}{0}$$

$$\Rightarrow x\,dx+y\,dy+z\,dz=0$$

Integrating we get

$$\int x\,dx+\int y\,dy+\int z\,dz=0$$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{a}{2}$$

$$x^2+y^2+z^2=c_1$$

Use Lagrange multipliers l, m, n we get

$$\frac{l\,dx+m\,dy+n\,dz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}$$

$$=\frac{ldx+mdy+n\,dz}{lmz-lny+mnx-lmz+lny-mnx}$$

$$=\frac{ldx+mdy+n\,dz}{0}$$

$$\Rightarrow l\,dx+m\,dy+n\,dz=0$$

Integrating we get

$$\int l\,dx+\int m\,dy+\int n\,dz=0$$

$$lx+my+nz=c_2$$

Hence the general solution is $\varphi(c_1,c_2)=0$

$$f(x^2+y^2+z^2,\ lx+my+nz)=0$$

26. Solve $(D^3-2D^2D')z=2e^{2x}+3x^2y$

Solution:

Given $(D^3-2D^2D')z=2e^{2x}+3x^2y$

The auxiliary equation is $m^3-2m^2=0$

$$m^2(m-2)=0$$

$$\Rightarrow \quad m=0,0,2$$

$$\therefore \quad C.F.=f_1(y)+xf_2(y)+f_3(y+2x)$$

$$P.I.=\frac{1}{(D^3-2D^2D')}(2e^{2x}+3x^2y)$$

$$P.I_1 = 2\frac{1}{D^3 - 2D^2D'}e^{2x}$$

replace $D = 2$ & $D' = 0$

$$= 2\frac{1}{8-0}e^{2x}$$

$$P.I_1 = \frac{1}{4}e^{2x}$$

$$P.I_2 = \frac{1}{D^3 - 2D^2D'}3x^2y$$

$$= \frac{1}{D^3\left(1 - \frac{2D'}{D}\right)}3x^2y$$

$$= \frac{3}{D^3}\left(1 - \frac{2D'}{D}\right)^{-1} x^2y$$

$$= \frac{3}{D^3}\left(1 + \frac{2D'}{D} + \left(\frac{2D'}{D}\right)^2 + \cdots\right)x^2y = \frac{3}{D^3}[x^2y + \frac{2D'(x^2y)}{D} + \frac{4D'^2(x^2y)}{D^2}$$

$$= \frac{3}{D^3}\left(x^2y + \frac{2}{D}x^2 + \frac{4}{D}(0) + \cdots\right) \qquad D'(x^2y) = x^2$$

$$= \frac{3}{D^3}\left(x^2y + \frac{2x^2}{3}\right) \qquad D'^2(x^2y)$$

$$= 0$$

$$= \frac{3}{D^2}\left(\frac{x^3y}{3} + \frac{2x^4}{12}\right)$$

$$= \frac{3}{D}\left(\frac{x^4y}{12} + \frac{2x^5}{60}\right)$$

$$= 3\left(\frac{x^5y}{60} + \frac{2x^6}{360}\right)$$

$$= \frac{x^5y}{20} + \frac{x^6}{60}$$

$$\therefore z = C.F. + P.I.$$

$$z = f_1(y) + xf_2(y) + f_3(y + 2x) + \frac{1}{4}e^{2x} + \frac{x^5y}{20} + \frac{x^6}{60}$$

27. Solve the equation $(D^2 - D'^2)z = e^{x-y}\sin(x + 2y)$

Solution:

Given $(D^2 - D'^2)z = e^{x-y}\sin(x + 2y)$

The auxiliary equation is $m^2 - 1 = 0$

$$\Rightarrow \qquad m = \pm 1$$

$$\therefore C.F. = f_1(y + x) + f_2(y - x)$$

$$P.I = \frac{1}{D^2 - D'^2}e^{x-y}\sin(x + 2y)$$

replace $D = (D + 1)$ & $D' = (D' - 1)$

$$P.I = e^{x-y}\frac{1}{(D + 1)^2 - (D' - 1)^2}\sin(x + 2y)$$

$$= e^{x-y}\frac{1}{D^2+2DD'+1-(D'^2-2DD'+1)}\sin(x+2y)$$

$$= e^{x-y}\frac{1}{D^2-D'^2+2D+2D'}\sin(x+2y)$$

$replace\ D^2 = -(1)^2 = -1\ \&\ D'^2 = -(2)^2 = -4$

$$= e^{x-y}\frac{1}{-1+4+2D+2D'}\sin(x+2y)$$

$$= e^{x-y}\frac{1}{3+2D+2D'}\sin(x+2y)$$

$$= e^{x-y}\frac{1}{2(D+D')+3}\sin(x+2y)$$

$$= e^{x-y}\frac{1}{2(D+D')+3}\ \frac{2(D+D')-3}{2(D+D')-3}\sin(x+2y)$$

$$= e^{x-y}\frac{2(D+D')-3}{[2(D+D')]^2-3^2}\ \sin(x+2y)$$

$$= e^{x-y}\frac{2D+2D'-3}{4[D^2+2DD'+D'^2]-9}\ \sin(x+2y)$$

$replace\ D^2\ by -1\ , D'^2\ by -4\ \ \&\ DD'\ by -2$

$$= e^{x-y}\frac{2D+2D'-3}{4[-1-4-4]-9}\ \sin(x+2y)$$

$$= e^{x-y}\frac{2D+2D'-3}{-45}\ \sin(x+2y)$$

$$= \frac{e^{x-y}}{-45}[2D\sin(x+2y)+2D'\sin(x+2y)-3\sin(x+2y)]$$

$$= \frac{e^{x-y}}{-45}[2\cos(x+2y)+4\cos(x+2y)-3\sin(x+2y)]$$

$$= \frac{e^{x-y}}{-45}[6\cos(x+2y)-3\sin(x+2y)]$$

$$= -\frac{e^{x-y}}{15}[2\cos(x+2y)-\sin(x+2y)]$$

Hence the solution is $z = C.F.+P.I$

$$z = f_1(y+x)\ + f_2(y-x) - \frac{e^{x-y}}{15}[2\cos(x+2y)-\sin(x+2y)]$$

28. Solve the equation $(D^3+D^2D'-4DD'^2-4D'^3)z=\cos(2x+y)$

Solution:

Given $(D^3+D^2D'-4DD'^2-4D'^3)z=\cos(2x+y)$

The auxiliary equation is

$m^3+m^2-4m-4=0$

$$m^2(m+1)-4(m+1)=0$$

$$(m^2-4)(m+1)=0$$

$$m=1,-2,2$$

$$\therefore C.F=f_1(y-x)+f_2(y-2x)+f_3(y+2x)$$

$$P.I=\frac{1}{\left(D^3+D^2D'-4DD'^2-4D'^3\right)}\cos(2x+y).$$

$$replace\ D^2=-(2)^2=-4, DD'=-2\ \&\ D'^2=-1$$

$$=\frac{1}{(-4D-4D'+4D+4D')}\cos(2x+y).$$

$$=\frac{1}{0}\cos(2x+$$
$y)$.

$$=x\frac{1}{(3D^2+2DD'-4D'^2)}\cos(2x+y).$$

$$replace\ D^2=-(2)^2=-4, DD'=-2\ \&\ D'^2=-1$$

$$=x\frac{1}{(-12-4+\ )}\cos(2x+y).$$

$$=x\frac{1}{(-12)}\cos(2x+y).$$

Hence the solution is $z=C.F.+P.I$

$$z=f_1(y-x)+f_2(y-2x)+f_3(y+2x)-\frac{x}{-12}\cos(2x+y)$$

29. Solve the equation $(D^2+2DD'+D'^2-2D-2D')z=\sin(x+2y)$

Solution:

Given $(D^2+2DD'+D'^2-2D-2D')z=\sin(x+2y)$

$$[(D+D')^2-2(D+D')]=\sin(x+2y)$$

$$ie.(D+D')(D+D'-2)z=\sin(x+2y)$$

To find the complementary function:

$$(D+D')(D+D'-2)z=0$$

$$if\ \ (D-m_1D'-c_1)(D-m_2D'-c_2)$$

$$then\ z=e^{c_1x}f_1(y+m_1x)+e^{c_2x}f_2(y+m_2x)$$

$$here\ \ m_1=-1, c_1=0$$

$$m_2=-1, c_2=2$$

$$\therefore C.F=e^{0x}f_1(y+(-1)x)+e^{2x}f_2(y+(-1)x)$$

$$C.F=f_1(y-x)+e^{2x}f_2(y-x)$$

To find the particular integral:

$$P.I.=\frac{1}{\left(D^2+2DD'+D'^2-2D-2D'\right)}\sin(x+2y)$$

$$=\frac{1}{(D+D')(D+D'-2)}\sin(x+2y)$$

$$=I.P\frac{1}{(D+D')(D+D'-2)}e^{i(x+2y)}$$

$$= I.P\frac{1}{(D+D')(D+D'-2)}e^{(ix+2i\ \ )}$$

$(replace\ D\ \ by\ i\ and\ D'\ by\ 2i\ \ )$

$$= I.P\frac{1}{(i+2i)(i+2i-2)}e^{(ix+2iy)}$$

$$= I.P\frac{1}{(3i)(3i-2)}e^{(ix+2\ \ \ )}$$

$$= I.P\frac{1}{3i}\ \frac{1}{(3i-2)}e^{(ix+2i\ \ )}$$

$$= I.P\frac{1}{3i}\ \frac{i}{i}\ \frac{3i+2}{(3i-2)(3i+2)}e^{(ix+2i\ \ )}$$

$$= I.P\frac{i}{-3}\ \frac{3i+2}{(-9-4)}e^{(ix+2i\ \ )}$$

$$= I.P\ \ \frac{-3+2i}{39}e^{(ix+2i\ \ )}$$

$$= I.P\ \ \frac{-3+2i}{39}[\cos(x+2y)+isin(x+2y)]$$

$$P.I = \frac{-3}{39}sin(x+2y)+\frac{2}{39}\cos(x+2y)$$

Hence the solution is $z = C.F.+P.I$

$$z = f_1(y-x)+e^{2x}f_2(y-x)-\frac{-3}{39}sin(x+2y)$$
$$+\frac{2}{39}\cos(x+2y)$$

30. Solve the equation $(D^2-2DD'+D'^2-3D+3D'+2)z=e^{2x-y}$

Solution:

$Given\quad (D^2-2DD'+D'^2-3D+3D'+2)z = e^{2x-y}$

This equation can be written as

$$(D-D'-1)(D-D'-2)z = 2e^{2x-y}$$

$$here\ m_1 = 1, c_1 = 1\ \&\ m_2 = 1, c_2 = 2$$

$$\therefore C.F. = e^x f_1(y+x)\ + e^{2x}f_2(y+x)$$

$$P.I. = \frac{1}{(D-D'-1)(D-D'-2)}e^{2x-y}$$

$replace\ D\ \ by\ \ 2\ \ \&\ \ \ D'\ \ by -1$

$$= \frac{1}{(2+1-1)(2+1-2)}e^{2x-y}$$

$$P.I. = \frac{1}{2}e^{2x-y}$$

$$\therefore z = C.F.+P.I.$$

$$z = e^x f_1(y+x)\ + e^{2x}f_2(y+x)+\tfrac{1}{2}e^{2x-y}$$

UNIT-II

PART A

1. State Dirichlet's condition for the convergence of the Fourier series of $f(x)$.

Solution:

If a function $f(x)$ is defined in (0.2π), it can be expressed as a Fourier series of the form $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx$

Where a_0, a_n, b_n are constants, provided the following conditions are satisfied

i) $f(x)$ is defined and single valued except possibly at a finite number of points in $(0,2\pi)$.
ii) $f(x)$ is periodic in $(0,2\pi)$.
iii) $f(x)$ *and* $f'(x)$ are piece wise continuous in$(0,2\pi)$.
iv) $f(x)$ has a finite number of maxima or minima in $(0,2\pi)$

2. Obtain the first term of the Fourier series for the function $f(x)=x^2,\ -\pi<x<\pi$

Solution:

Given $f(x)=x^2$ *in* $(-\pi,\pi)$

$$f(-x)=(-x)^2=x^2=f(x)$$

$$f(-x)=f(x)$$

Therefore $f(x)$ is an even function

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)dx \;=\frac{2}{\pi}\int_0^{\pi} x^2dx$$

$$=\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\frac{\pi^3}{3}-0\right]=\frac{2\pi^2}{3}$$

$$a_0=\frac{2\pi^2}{3}$$

3. If $x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$,deduce that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots=\frac{\pi^2}{6}$.

Solution:

Given $x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \ldots\ldots\ldots\ldots(1)$

$$=\frac{\pi^2}{3}+4\left[\frac{-\cos x}{1^2}+\frac{\cos 2x}{2^2}-\frac{\cos 3x}{3^2}+\cdots\ldots\right]$$

$$=\frac{\pi^2}{3}-4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\cdots\ldots\right]$$

The given Fourier series is $f(x)=x^2$ *is an even function in the interval,* $-\pi<x<\pi$.

$\therefore$ *put* $x=\pi$ *in* (1)

Here $x=\pi$ in a point of discontinuity.

$$\frac{f(\pi)+f(-\pi)}{2}=\frac{\pi^2}{3}-4\left[\frac{\cos\pi}{1^2}-\frac{\cos 2\pi}{2^2}+\frac{\cos 3\pi}{3^2}-\cdots\ldots\right]$$

$$\frac{\pi^2+\pi^2}{2}=\frac{\pi^2}{3}-4\left[\frac{-1}{1^2}-\frac{1}{2^2}+\frac{-1}{3^2}-\cdots\ldots.\right]$$

$$\frac{2\pi^2}{2}=\frac{\pi^2}{3}+4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\ldots.\right]$$

$$\pi^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$4\sum_{n=1}^{\infty}\frac{1}{n^2}=\pi^2-\frac{\pi^2}{3}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{4}\left[\frac{3\pi^2-\pi^2}{3}\right]$$

$$\therefore\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\frac{\pi^2}{6}$$

4. Interpret the constant term in the expansion for $\cos^2 x$ as a Fourier series in the interval $(-\pi,\pi)$.

Solution:

Given $f(x)=cos^2x=\frac{1+\cos}{2}$

Hence $f(x)$ is an even function.

Constant term is $a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx$

$$a_0=\frac{2}{\pi}\int_0^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx \quad =\frac{2}{2\pi}\int_0^{\pi}(1+\cos 2x)dx$$

$$=\frac{1}{\pi}\left[x+\frac{\sin 2x}{2}\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[\pi+\frac{\sin 2\pi}{2}-0-\frac{\sin 0}{2}\right]$$

$$a_0=\frac{1}{\pi}(\pi)=1$$

$$a_0=1$$

5. Find the root mean square value of $f(x)=e^{-ax},\ a>0$

Solution:

The root mean square value is $[\bar{y}]^2=\frac{1}{b-a}\int_a^b[f(x)]^2\,dx$

$$[\bar{y}]^2=\frac{1}{l}\int_0^l[f(x)]^2\,dx$$

$$[\bar{y}]^2=\frac{1}{l}\int_0^l(e^{-ax})^2\,dx=\frac{1}{l}\left[\frac{e^{-2ax}}{-2a}\right]_0^l=\frac{1}{l}\left[\frac{e^{-2al}}{-2a}-\frac{e^0}{-2a}\right]$$

$$[\bar{y}]^2=\frac{1}{2al}[1-e^{-2al}]$$

6. Give the expression for the Fourier series coefficient b_n for the function $f(x)$ defined in $(-2,2)$.

Solution:

$$b_n=0$$

7. Without finding the values of $a_0,\ a_n\ \&\ b_n$, the Fourier coefficients of Fourier series for the function $f(x)=x^2$ in the interval $(0,\ \pi)$. Find the value $\frac{a_0^2}{2}+\sum_{n=1}^{\infty}a_n^2+b_n^2$.

Solution:

$$\frac{1}{\pi}\int_0^{\pi}[f(x)]^2\ dx=\frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+\ b_n^2).\ldots\ldots(1)$$

$$(1)\times 2\quad\Rightarrow\quad\frac{2}{\pi}\int_0^{\pi}[f(x)]^2\ dx$$

$$=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+\ b_n^2).\ldots\ldots(1)$$

$$given\ f(x)=x^2$$

$$\therefore\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+\ b_n^2)=\frac{2}{\pi}\int_0^{\pi}[x^2]^2\ dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x^4\ dx=\frac{2}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi}\quad=\frac{2}{\pi}\left[\frac{\pi^5}{5}-0\right]\quad=\frac{2\pi^4}{5}$$

8. Define the root mean square value of $f(x)$ over the interval $(a,\ b)$.

Solution:

Let $f(x)$ be a function defined in an interval (a, b) then

$$\bar{Y}=\sqrt{\frac{\int_a^b[f(x)]^2dx}{b-a}}$$

$$[\bar{Y}]^2=\frac{1}{b-a}\int_a^b[f(x)]^2\,dx$$

9. Find the co efficient b_n of the Fourier series for the function $f(x)=x\sin x$ in the interval $(-2,\ 2)$.

Solution:

Given $f(x)=x\sin x$ is an even function $\quad\therefore\ b_n=0$

10. Find the value of a_0 in the Fourier series expansion of $f(x)=e^x$ in the interval $(0,\ 2\pi)$.

Solution:

We know that $a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx$

$$\therefore\ a_0=\frac{1}{\pi}\int_0^{2\pi}e^x\,dx\quad=\frac{1}{\pi}[e^x]_0^{2\pi}$$

$$=\frac{1}{\pi}[e^{2\pi}-e^0]$$

$$\therefore\ a_0=\frac{e^{2\pi}-1}{\pi}$$

11. Expand $f(x)=1$ as a half range sine series in the interval $(0,\ \pi)$.

Solution:

Given $f(x) = 1$ in $(0,\pi)$

The half range Fourier Sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \cdots\cdots\cdots (1)$$

Where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$

$$= \frac{2}{\pi}\int_0^{\pi} (1)\sin nx\, dx$$

$$= \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{-2}{\pi n}[\cos n\pi - \cos 0]$$

$$= -\frac{2}{n\pi}[(-1)^n - 1]$$

$$= \frac{2}{n\pi}[1-(-1)^n]$$

$$\therefore b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$\Rightarrow f(x) = \sum_{n=odd}^{\infty} \frac{4}{n\pi}\sin nx = \frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n}\sin nx$$

12. If $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$. Deduce that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6}$.

Solution:

Given $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \ldots\ldots\ldots\ldots (1)$

$$= \frac{\pi^2}{3} + 4\left[\frac{-\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots \ldots\right]$$

$$= \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots \ldots\right]$$

The given Fourier series is $f(x) =$ x^2 *is an even function in the intervel*, $-\pi < x < \pi$.

$\therefore$ *put* $x = \pi$ *in* (1)

Here $x = \pi$ in a point of discontinuity.

$$\frac{f(\pi)+f(-\pi)}{2} = \frac{\pi^2}{3} - 4\left[\frac{\cos\pi}{1^2} - \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} - \cdots \ldots\right]$$

$$\frac{\pi^2+\pi^2}{2} = \frac{\pi^2}{3} - 4\left[\frac{-1}{1^2} - \frac{1}{2^2} + \frac{-1}{3^2} - \cdots \ldots\right]$$

$$\frac{2\pi^2}{2} = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \ldots\right]$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$4\sum_{n=1}^{\infty}\frac{1}{n^2} = \pi^2 - \frac{\pi^2}{3}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{4}\left[\frac{3\pi^2-\pi^2}{3}\right]$$

$$\therefore \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\ldots\ldots\ldots.=\frac{\pi^2}{6}$$

13. Find the root mean square value of the function $f(x)=x$ in the interval $(0,l)$

Solution:

The root mean square value is $[\bar{y}]^2=\frac{1}{b-a}\int_a^b[f(x)]^2\,dx$

$$[\bar{y}]^2=\frac{1}{l}\int_0^l[f(x)]^2\,dx$$

$$[\bar{y}]^2=\frac{1}{l}\int_0^l x^2\,dx=\frac{1}{l}\left[\frac{x^3}{3}\right]_0^l=\frac{1}{l}\left[\frac{l^3}{3}-0\right]$$

$$[\bar{y}]^2=\frac{l^2}{2}$$

$$\bar{y}=\frac{l}{\sqrt{2}}$$

14. Define Harmonic analysis.

Solution:

The process of finding the Fourier series for a function given by numerical values known as harmonic analysis.

15. Find half range sine series for $f(x)=k\ ,0<x<\pi$.

Solution:

Given $f(x)=1$ in $(0,\pi)$

The half range Fourier Sine series is given by

$$f(x)=\sum_{n=1}^{\infty}b_n\ \sin nx \cdots\cdots\cdots\cdots (1)$$

Where $b_n=\frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx$

$$=\frac{2}{\pi}\int_0^\pi (k)\sin nx\,dx$$

$$=\frac{2k}{\pi}\left[-\frac{\cos nx}{n}\right]_0^\pi$$

$$=\frac{-2k}{\pi n}[\cos n\pi-\cos 0]$$

$$=-\frac{2k}{n\pi}[(-1)^n-1]$$

$$=\frac{2k}{n\pi}[1-(-1)^n]$$

$$\therefore b_n=\begin{cases}0 & if\quad n\ is\ \ even\\ \frac{4k}{n\pi} & if\ n\ is\ \ odd\end{cases}$$

$$\Rightarrow f(x)=\sum_{n=odd}^{\infty}\frac{4k}{n\pi}\sin nx=\frac{4k}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n}\sin nx$$

PART B

16. Find the Fourier series expansion for $f(x)=x^2$ in $[0,2\pi]$ and periodic with period 2π. Hence deduce that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

Solution:

Given $f(x)=x^2$ in $(0,2\pi)$

The Fourier series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

Where $a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}x^2dx$$

$$=\frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}=\frac{1}{\pi}\left[\frac{8\pi^3}{3}\right]$$

$$a_0=\frac{8\pi^2}{3}$$

W.K.T $a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}x^2\cos nx\,dx$$

using Bernouille's formula $\int uv\,dx=uv_1-u'v_2+u''v_3-\cdots$

$$=\frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)-2x\left(\frac{-\cos nx}{n^2}\right)+(2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[0+\frac{4\pi}{n^2}-0-(0)\right]$$

$$=\frac{1}{\pi}\left(\frac{4\pi}{n^2}\right)=\frac{4}{n^2}\quad\Rightarrow\quad a_n=\frac{4}{n^2}$$

W.K.T $b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}(\pi-x)^2\sin nx\,dx$$

using Bernouille's formula $\int uv\,dx=uv_1-u'v_2+u''v_3-\cdots$

$$=\frac{1}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right)-2x\left(\frac{-\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[4\pi^2\left(\frac{-\cos 2\pi n}{n}\right)+0+2\left(\frac{\cos 2\pi n}{n^3}\right)-0+0-2\left(\frac{\cos 0}{n^3}\right)\right]$$

$$=\frac{1}{\pi}\left[-\frac{4\pi^2}{n}+0+\frac{2}{n^3}+0+0-\frac{2}{n^3}\right]=0$$

$$b_n=-\frac{4\pi}{n}$$

Substituting a_0, a_n and b_n values in equation (1) we get

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$f(x)=\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx-4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$(x)^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx - 4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

Here $x = 0$ in a point of continuity.

$$\frac{f(0)+f(2\pi)}{2} = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos 0$$

$$\frac{0+4\pi^2}{2} = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$2\pi^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$2\,\pi^2 - \frac{4\pi^2}{3} = 4\left[\sum_{n=1}^{\infty}\frac{1}{n^2}\right]$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

17. Evaluate the Fourier series expansion of $f(x)=\begin{cases} x, & 0\le x\le \pi \\ 2\pi - x, & \pi \le x \le 2\pi\end{cases}$**. Deduce that** $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots = \frac{\pi^2}{8}$

Solution:

Given $f(x)=\begin{cases} x, & 0\le x\le \pi \\ 2\pi - x, & \pi \le x \le 2\pi\end{cases}$

The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{1}{\pi}\left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - 0 + 4\pi^2 - \frac{4\pi^2}{2} - 2\pi^2 + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 4\pi^2 - 2\pi^2 - 2\pi^2 + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2}+\frac{\pi^2}{2}\right] \qquad = \frac{1}{\pi}\left[\frac{2\pi^2}{2}\right] \qquad \therefore a_0 = \pi$$

$u = x,$ $u = 2\pi - x$	$v = \cos nx$
$u' = 1,$ $u' = -1$	$v_1 = \frac{\sin nx}{n}$
$u'' = 0$	$v_2 = \frac{-\cos nx}{n^2}$

$u = x,$ $u = 2\pi - x$	$v = \sin nx$
$u' = 1,$ $u' = -1$	$v_1 = \frac{-\cos nx}{n}$
$u'' = 0$	$v_2 = \frac{-\sin nx}{n^2}$

W.K.T $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$

$$= \frac{1}{\pi}\int_0^{\pi} x\cos nx dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\cos nx\, dx$$

using Bernouille's formula $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots\cdots$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos nx}{n^2}\right)\right]_0^{\pi} + \left[\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(\left(\frac{-\cos n\pi}{n^2}\right) - \frac{-\cos 0}{n^2}\right) + \left(\frac{\cos 2n\pi}{n^2} - \frac{-\cos n\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right]$$

$[\because \sin n\pi = 0, \cos n\pi = (-1)^n, \sin 2n\pi = 0, cos\ 2n\pi = 1, \cos 0 = 1 \& \sin 0 = 0]$

$$\therefore a_n = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$$\therefore a_n = \begin{cases} 0, & when\ n\ is\ even \\ \frac{-4}{n^2\pi}, & when\ n\ is\ odd \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin nx dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\sin nx\, dx$$

using Bernouille's formula $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots\cdots$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{-\cos}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{-\cos nx}{n}\right)\right]_{\pi}^{2\pi}$$

$$=\frac{1}{\pi}\left[\left(\pi\left(\frac{-\cos n\pi}{n}\right)-0\right)+\left(0-\pi\left(\frac{-\cos n\pi}{n}\right)\right)\right]$$

$[\because \sin n\pi=0, \cos n\pi=(-1)^n, \sin 2n\pi=0, \cos 2n\pi=1, \cos 0=1 \,\&\, \sin 0=0]$

$$=\frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n^2}+\pi\frac{(-1)^n}{n^2}\right]$$

$\therefore b_n=0$

substituting $a_0, a_n \& b_n$ *values in* (1) *we get*

$$f(x)=\frac{\pi}{2}+\sum_{n=odd}^{\infty}\frac{-4}{n^2\pi}\cos nx+\sum_{n=1}^{\infty}(0)\sin nx$$

$$f(x)=\frac{\pi}{2}+\sum_{n=odd}^{\infty}\frac{-4}{n^2\pi}\cos nx \ldots\ldots\ldots\ldots\ldots(2)$$

Deduction:

Let $x=0$

Here $x=0$ in a point of continuity

$$f(0)=\frac{\pi}{2}+\sum_{n=odd}-\frac{4}{n^2\pi}\cos 0$$

$$0=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=odd}\frac{1}{n^2}$$

$$-\frac{\pi}{2}=-\frac{4}{\pi}\sum_{n=odd}\frac{1}{n^2}$$

$$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\cdots$$

18. Find the Fourier series for $f(x)=2x-x^2$ in $0<x<2$.

Solution:

Given $f(x)=2x-x^2$ in $(0,2)$

We know that the Fourier series in the interval $(o,2l)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\ldots\ldots(1)$$

Here $2l=2, l=1$

$$a_0=\frac{1}{l}\int_0^{2l}f(x)dx$$

$$=\int_0^2 f(x)dx$$

$$=\int_0^2(2x-x^2)dx$$

$$= \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2$$

$$= \frac{8}{2} - \frac{8}{3} = 4 - \frac{8}{3}$$

$$a_0 = \frac{4}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\, cos\frac{n\pi x}{l}\, dx$$

$$a_n = \int_0^2 f(x)\ \cos n\pi x\ dx$$

$$a_n = \int_0^2 f(x)\ \cos n\pi x\ dx$$

$u = (2x - x^2)$	$v = \cos n\pi x$
$u' = (2 - 2x)$	$v_1 = \frac{\sin n\pi x}{n\pi}$
$u'' = (-2)$	$v_2 = \frac{-\cos n\pi x}{n^2\pi^2}$
$u''' = 0$	$v_3 = \frac{-\sin nx}{n^3\pi^{3.}}$

$$a_n = \int_0^2 (2x - x^2)\ \cos n\pi x\ dx$$

$using\ Bernouille's\ formula\ \int uv\, dx\ = uv_1 - u'v_2 + u''\, v_3 - \cdots\cdots$

$$= \left[(2x - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (2 - 2x)\left(\frac{-\cos\ n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$= \left[(2 - 2x)\left(\frac{\cos\ n\pi x}{n^2\pi^2}\right)\right]_0^2$$

$$= \left[\left(-2\frac{\cos\ 2n\pi}{n^2\pi^2}\right) - \left(2\frac{\cos\ 0}{n^2\pi^2}\right)\right]$$

$$= \left[-\frac{2}{n^2\pi^2} - \frac{2}{n^2\pi^2}\right]$$

$[\because \cos 2n\pi = 1; \sin 2n\pi = 0; \cos 0 = 1\ \&\ \sin 0 = 0]$

$$a_n = -\frac{4}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\, sin\frac{n\pi x}{l}\, dx$$

$$b_n = \int_0^2 f(x)\ \sin n\pi x\ dx$$

$u = (2x - x^2)$	$v = \sin n\pi x$
$u' = (1 + 2x)$	$v_1 = \frac{-\cos\ n\pi x}{n\pi}$
$u'' = 2$	$v_2 = \frac{-\sin n\pi x}{n^2\pi^2}$
$u''' = 0$	$v_3 = \frac{\cos n\pi x}{n^3\pi^{3.}}$

$$b_n = \int_0^2 (2x - x^2)\ \sin n\pi x\ dx$$

using Bernouille's formula $\int \mathrm{uv\ dx} = \mathrm{uv_1 - u'v_2 + u''\, v_3} - \cdots\cdots$

$$= \left[(2x - x^2)\left(\frac{-\cos\ n\pi x}{n\pi}\right) - (2 - 2x)\left(\frac{-\sin\ n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$= \left[(2x - x^2)\left(\frac{-\cos\ n\pi x}{n\pi}\right) + (-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$= \left[\left(0 - 2\frac{\cos 2n\pi}{n^3\pi^3}\right) - \left(0 - \frac{2}{n^3\pi^3}\right)\right]$$

$$= \left[-\frac{2}{n^3\pi^3} + \frac{2}{n^3\pi^3}\right]$$

$b_n = 0$

Substituting a_0, a_n and b_n values in equation (1) we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$f(x) = \frac{4}{3*2} + \sum_{n=1}^{\infty}\left(-\frac{4}{n^2\pi^2}\right)\cos n\pi x + \sum_{n=1}^{\infty}(0)\sin n\pi x$$

$$f(x) = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$$

19. Find the Fourier series of periodicity 3 for $f(x) = 2x - x^2$ in $0 < x < 3$.

Solution:

Given $f(x) = 2x - x^2$ in (0,3)

We know that the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Here $2l = 3 \quad \Rightarrow l = \frac{3}{2} \quad \Rightarrow \frac{1}{l} = \frac{2}{3}$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$$

$$= \frac{2}{3}\int_0^3 f(x)dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)dx$$

$$= \frac{2}{3}\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{2}{3}\left[\frac{18}{2} - \frac{27}{3}\right]$$

$$= \frac{2}{3}[9 - 9] = 0$$

$$\therefore a_0 = 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{3}\int_0^3 f(x)\cos\frac{2n\pi x}{3}dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}dx$$

$u = (2x - x^2)$	$v = \cos\frac{2n\pi x}{3}$
$u' = (2 - 2x)$	$v_1 = \frac{\sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}$
$u'' = (-2)$	$v_3 = \frac{-\cos\frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}}$
$u''' = 0$	$v_3 = \frac{-\sin\frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}}$

using Bernouille's formula $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$

$$\frac{2}{3}\left[(2x - x^2)\left(\frac{\sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right) - (2 - 2x)\left(\frac{-\cos\frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}}\right) + (-2)\left(\frac{-\sin\frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}}\right)\right]_0^3$$

$$= \frac{2}{3}\left[-(2-2x)\left(\frac{-\cos\frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}}\right)\right]_0^3$$

$$= \frac{2}{3}\left[-4\left(\frac{\cos 2n\pi}{\frac{4n^2\pi^2}{9}}\right) - 2\frac{\cos 0}{\frac{4n^2\pi^2}{9}}\right]$$

$$= \frac{2}{3}\left[-\frac{9}{n^2\pi^2} - \frac{9}{2n^2\pi^2}\right]$$

$[\because \cos 2n\pi = 1; \sin 2n\pi = 0; \cos 0 = 1 \,\&\, \sin 0 = 0]$

$$= \frac{2}{3}\left[\frac{-18-9}{2n^2\pi^2}\right]$$

$$= \frac{2}{3}\left[\frac{-27}{2n^2\pi^2}\right] = \frac{-9}{n^2\pi^2}$$

$$\therefore\ a_n = \frac{-9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx\ ;\ b_n = \frac{2}{3}\int_0^3 f(x)\sin\frac{2n\pi x}{3}dx$$

$$b_n = \frac{2}{3}\int_0^3 (2x-x^2)\sin\frac{2n\pi x}{3}dx$$

using Bernouille's formula $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$

$u = (2x-x^2)$	$v = \sin\frac{2n\pi x}{3}$
$u' = (2-2x)$	$= \frac{-\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}$
$u'' = (-2)$	$= \frac{-\sin\frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}}$
$u''' = 0$	$= \frac{\cos\frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}}$

$$= \frac{2}{3}\left[(2x-x^2)\left(\frac{-\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right) - (2-2x)\left(\frac{-\sin\frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}}\right) + (-2)\left(\frac{\cos\frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}}\right)\right]_0^3$$

$$= \frac{2}{3}\left[(2x-x^2)\left(\frac{-\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right) + (-2)\left(\frac{\cos\frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\left(-3\left(\frac{-\cos 2n\pi}{\frac{2n\pi}{3}}\right) + (-2)\frac{\cos 2n\pi}{\frac{8n^3\pi^3}{27}}\right) - \left(0 - 2\frac{\cos 0}{\frac{8n^3\pi^3}{27}}\right)\right]$$

$$= \frac{2}{3}\left[\frac{9}{2n\pi} - 2\left(\frac{1}{\frac{8n^3\pi^3}{27}}\right) + 2\left(\frac{1}{\frac{8n^3\pi^3}{27}}\right)\right]$$

$[\because \cos 2n\pi = 1; \sin 2n\pi = 0; \cos 0 = 1 \,\&\, \sin 0 = 0]$

$$= \frac{2}{3}\left[\frac{9}{2n\pi}\right]$$

$$\therefore b_n = \frac{3}{n\pi}$$

Substituting a_0, a_n and b_n values in equation (1) we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{3}$$

$$f(x) = 0 + \sum_{n=1}^{\infty} \frac{-9}{n^2\pi^2} cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} \frac{3}{n\pi} sin\frac{2n\pi x}{3}$$

$$f(x) = \frac{-9}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} cos\frac{2n\pi x}{3} + \frac{3}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} sin\frac{2n\pi x}{3}$$

20. Obtain the Fourier series for the function $f(x)$ given by $f(x) = \begin{cases} 1-x, -\pi < x < 0 \\ 1+x, 0 < x < \pi \end{cases}$ .Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots = \frac{\pi^2}{8}$.

Solution:

Given $f(x) = \begin{cases} 1-x, & -\pi < x < 0 \\ 1+x, & 0 < x < \pi \end{cases}$

$$let \quad f(x) = \begin{cases} \phi_1(x), & -\pi < x < 0 \\ \phi_2(x), & 0 < x < \pi \end{cases}$$

$$where \ \ \phi_1(x) = 1 + x; \phi_2(x) = 1 - x$$

$$\phi_1(-x) = 1 - x = \phi_2(x)$$

$\therefore f(x)$ is an even function

hence $b_n = 0$

Then the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

Now $a_0 = \frac{2}{\pi}\int_0^{\pi} (1+x)\,dx = \frac{2}{\pi}\left[x + \frac{x^2}{2}\right]_0^{\pi}$

$$= \frac{2}{\pi}\left[\left(\pi + \frac{\pi^2}{2}\right) - (0+0)\right] = \frac{2}{\pi}\left[\frac{2\pi + \pi^2}{2} - 0\right]$$

$$= \frac{2\pi}{\pi}\left[\frac{2+\pi}{2}\right]$$

$$\therefore a_0 = 2 + \pi$$

$u = (1+x)$	$v = \cos nx$
$u' = 1,$	$v_1 = \frac{\sin nx}{n}$
$u'' = 0$	$v_2 = \frac{-\cos nx}{n^2}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) cosnx\,dx$$

$$= \int_0^{\pi} (1+x) cosnx\,dx$$

using Bernouille's formula $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \ldots\ldots$

$$= \frac{2}{\pi}\left[(1+x)\left(\frac{sinnx}{n}\right) - \left(\frac{-cosnx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{cosnx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{cosn\pi}{n^2}\right) - \left(\frac{cos0}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$=\frac{2}{\pi}\left[\frac{(-1)^n}{n^2}-\frac{1}{n^2}\right]$$

$$a_n=\frac{2}{n^2\pi}[(-1)^n-1]$$

$$\therefore a_n=\begin{cases}\frac{-4}{n^2\pi} & if\ n\ \ is\ \ odd\\ 0 & if\ \ n\ is\ even\end{cases}$$

$now\ \ (1)\ \Rightarrow$

$$f(x)=\frac{2+\pi}{2}+\sum_{n=1}^{\infty}\frac{-4}{n^2\pi}\cos nx$$

$$f(x)=\frac{\pi+2}{2}-\frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n^2}\cos nx\ \ \dots\dots\dots(2)$$

which is the required Fourier series

put x = 0 is a point of discontinuity in the middle point.

$$\therefore \frac{f(0+)+f(0-)}{2}=\frac{(1-0)+(1+0)}{2}=\frac{2}{2}$$

$$=1 \qquad since\ f(0)=1$$

$$\therefore (2)\Rightarrow \qquad f(0)=\frac{\pi+2}{2}-\frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow 1=\frac{\pi+2}{2}-\frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n^2}=\frac{\pi+2}{2}-1$$

$$\Rightarrow \frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n^2}=\frac{\pi+2-2}{2}$$

$$\Rightarrow \frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n^2}=\frac{\pi}{2}$$

$$\Rightarrow \sum_{n=odd}^{\infty}\frac{1}{n^2}=\frac{\pi}{2}*\frac{\pi}{4}$$

$$\Rightarrow \sum_{n=odd}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{8}$$

$$\Rightarrow \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\dots\dots.=\frac{\pi^2}{8}$$

21. **Obtain the Fourier series to represent the function** $f(x)=|x|$ **in** $-\pi<x<\pi$ **and deduce** $\sum_{1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$**.**

Solution:

Given $f(x)=|x|\ \ -\pi<x<\pi$.

$$f(-x)=|-x|=|x|$$

$$f(-x)=f(x)$$

$\therefore f(x)$is an even function.

hence $b_n=0$

Then the required fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} |x|dx = \frac{2}{\pi}\int_0^{\pi} xdx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^2 - 0}{2}\right] = \frac{2}{\pi}\frac{\pi^2}{2}$$

$\therefore a_0 = \pi$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

using Bernouille's formula $\int uv\, dx = uv_1 - u'v_2 + u''v_3 - \cdots$

$$a_n = \frac{2}{\pi}\int_0^{\pi} |x|\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\, dx$$

$u = x,$	$v = \cos nx$
$u' = 1,$	$v_1 = \frac{\sin nx}{n}$
$u'' = 0$	$v_2 = \frac{-\cos nx}{n^2}$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{\cos n\pi}{n^2}\right) - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{\cos n\pi}{n^2}\right) - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{(-1)^n}{n^2}\right) - \frac{1}{n^2}\right]$$

$\because \sin n\pi = 0 \,\&\, \cos n\pi = (-1)^n$

$$= \frac{2}{n^2\pi}[(-1)^n - 1]$$

$$\therefore a_n = \begin{cases} 0, & when\ n\ is\ even \\ \frac{-4}{n^2\pi}, & when\ n\ is\ odd \end{cases}$$

substituting $a_0, a_n \& b_n$ values in (1)we get

$$f(x) = \frac{\pi}{2} + \sum_{n=odd}^{\infty} \frac{-4}{n^2\pi}\cos nx$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=odd}^{\infty} \frac{1}{n^2}\cos nx \ldots\ldots\ldots\ldots\ldots (2)$$

<u>Deduction:</u>

Here $x = 0$ is a point of continuity .

$$f(x) = |x| \qquad f(0) = 0$$

put x = 0 in (2) we get

$$f(0)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n^2}\cos 0$$

$$0=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n^2}$$

$$\frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n^2}=\frac{\pi}{2} \qquad \sum_{n=odd}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{8}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$$

$$ie,\quad \frac{1}{1^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{8}$$

22. Find the Fourier series expansion for $f(x)=x^2$ in $(-\pi,\pi)$ and hence deduce that $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots=\frac{\pi^4}{90}$.

Solution:

Given $f(x)=x^2$ is an even function

∴ Fourier series expansion is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\, cosnx$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x^2\,dx$$

$$=\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} \qquad =\frac{2}{\pi}\left[\frac{\pi^3}{3}-0\right]$$

$$a_0=\frac{2\pi^2}{3}$$

w.k.t $a_n=\frac{2}{\pi}\int_0^{\pi}x^2\cos nx\,dx$

using Bernouille's formula

$u=x^2$	$v=\cos nx$
$u'=2x$	$v_1=\frac{\sin\ nx}{n}$
$u''=2$	$v_2=\frac{-\cos\ nx}{n^2}$
$u''' = 0$	$v_3=\frac{-\sin\ nx}{n^3}$

$$=\frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)-2x\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[2x\left(\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n^2}\right]$$

$$=\frac{4}{n^2}(-1)^n \qquad [\cos n\pi=(-1)^n]$$

The required Fourier series be

$$f(x)=\frac{\frac{2\pi^2}{3}}{2}+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\ cosnx$$

$$=\frac{\pi^2}{3}+4\left[\frac{-\cos x}{1^2}+\frac{\cos 2x}{2^2}-\frac{\cos 3x}{3^2}+\cdots\cdots\right]$$

$$=\frac{\pi^2}{3}-4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\cdots\cdots\right]$$

By Parseval's identity $\frac{{a_0}^2}{2}+\sum_{n=`1}^{\infty}{a_n}^2=\frac{2}{\pi}\int_0^{\pi}(f(x))^2dx$

$$\frac{4\pi^4}{\frac{9}{2}} + 16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{2}{\pi}\int_0^{\pi} x^4 dx$$

$$\frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{2}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi} = 16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{2\pi^4}{5} - \frac{2\pi^4}{9} = \frac{2\pi^4(9-5)}{45} = \frac{8\pi^4}{45}$$

$$\Rightarrow \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$$

23. Find the Fourier series expansion of $f(x) = 1 - x^2$ in $(-1,1)$.

Solution:

Given $f(x) = 1 - x^2$ in $(-1,1)$

$f(-x) = 1 - (-x)^2 = 1 - x^2 = f(x)$

$\therefore f(x)$ is an even function in $(-1,1)$

Then the fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

here $l = 1$ $\therefore$ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \ldots\ldots\ldots(1)$

where $a_0 = \frac{2}{l}\int_0^l f(x)dx$

here $l = 1 \Rightarrow a_0 = \frac{2}{1}\int_0^1 (1 - x^2)dx$

$$= 2\left[x - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\left(1 - \frac{1}{3}\right) - 0\right] = 2\left(\frac{2}{3}\right)$$

$$\therefore a_0 = \frac{4}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos n\pi x\,dx$$

$$= \frac{2}{l}\int_0^l (1 - x^2)\cos n\pi x\,dx$$

$u = 1 - x^2$	$v = \cos n\pi x$
$u' = -2x$	$v_1 = \frac{\sin n\pi x}{n\pi}$
$u'' = -2$	$v_2 = \frac{-\cos n\pi x}{n^2\pi^2}$
$u''' = 0$	$v_3 = \frac{-\sin n\pi x}{n^3\pi^3}$

using Bernouille's formula $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots\cdots$

$$= 2\left[(1 - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[-2x\left(\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= 2\left[-2\frac{\cos n\pi}{n^2\pi^2} - 0\right] \quad a_n = -4\frac{(-1)^n}{n^2\pi^2}$$

sub the values of a_0 & a_n in (1) we get

$$f(x)=\frac{4}{3*2}+\sum_{n=1}^{\infty}-4\frac{(-1)^n}{n^2\pi^2}cos\frac{n\pi x}{l}$$

$$f(x)=\frac{2}{3}-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}cos\, n\pi x$$

24. Obtain the half range cosine series for $f(x)=x$ in $(0,\pi)$.

Solution:

Given $f(x)=x\ in\ (0,\pi)$

The Half Range Fourier cosine series is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\, cosnx\ \dots\dots\dots\dots\dots(1)$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x\,dx=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}\qquad=\frac{2}{\pi}\left[\frac{\pi^2}{2}-0\right]$$

$$=\frac{2}{\pi}\frac{\pi^2}{2}$$

$$\therefore a_0=\pi$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx$$

$u=x$	$v=\cos nx$
$u'=1,$	$v_1=\frac{\sin nx}{n}$
$u''=0$	$v_2=\frac{-\cos nx}{n^2}$

using Bernouille's formula

$\int uv\,dx=uv_1-u'v_2+u''v_3-\cdots$

$$=\frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right)-\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\left(\frac{\cos nx}{n^2}\right)\right]_0^{\pi}\qquad=\frac{2}{\pi}\left[\frac{\cos n\pi}{n^2}-\frac{1}{n^2}\right]$$

$$=\frac{2}{n^2\pi}[(-1)^n-1]$$

$$\therefore\ a_n=\frac{2}{n^2\pi}[(-1)^n-1]$$

$$\therefore\ a_n=\begin{cases}0 & ;\ when\ n\ is\ even\\ -\frac{4}{n^2\pi}; & when\ n is\ odd\end{cases}$$

Substituting a_0 & a_n values in equation (1) we get

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\, cosnx$$

$$f(x)=\frac{\pi}{2}+\sum_{n=1,3,5....}^{\infty}-\frac{4}{n^2\pi}cosn\pi x$$

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{\cos x}{n^2}+\frac{\cos 3x}{n^2}+\frac{\cos 5x}{n^2}+\cdots .....\right]$$

25. Obtain the half range sine series of $f(x)=lx-x^2$ in (o,l).

Solution:

The Fourier sine series in $(0,l)$ is given by

$$f(x)=\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l} \text{ where } b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{l}\int_0^l (lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

using Bernouille's formula $\int uv\,dx = uv_1-u'v_2+u''v_3-\cdots$

$$=\frac{2}{l}\left[(lx-x^2)\frac{l}{n\pi}\frac{-cosn\pi x}{l}-(l-2x)\frac{l^2}{n^2\pi^2}\frac{-\sin n\pi}{l}x-2\frac{l^3}{n^3\pi^3}\frac{cosn\pi x}{l}\right]_0^l$$

$$=\frac{2}{l}\left[-(lx-x^2)\frac{l}{n\pi}\frac{cosn\pi x}{l}-2\frac{l^3}{n^3\pi^3}\frac{cosn\pi x}{l}\right]_0^l$$

$$=\frac{2}{l}\left[\left(0-\frac{2l^3}{n^3\pi^3}\cos\frac{n\pi l}{l}\right)-\left(0-\frac{2l^3}{n^3\pi^3}\cos 0\right)\right]$$

$$=\frac{2}{l}\left[-\frac{2l^3}{n^3\pi^3}(-1)^n+\frac{2l^3}{n^3\pi^3}\right]$$

$$=\frac{2}{l}\left[\frac{2l^3}{n^3\pi^3}(1-(-1)^n)\right]$$

$$=\frac{4}{l}\left[\frac{l^3}{n^3\pi^3}(1-(-1)^n)\right]$$

$$=\begin{cases}0, if\ n\ is\ even\\ \frac{8l^2}{n^3\pi^3}, if\ n\ is\ odd\end{cases}$$

$$f(x)=\sum_{n=0dd}^{\infty}\frac{8l^2}{n^3\pi^3}\sin\frac{n\pi x}{l}$$

$u=(lx-x^2)$	$v=sin\frac{n\pi x}{L}$
$u'=(l-2x)$	$v_1=\frac{-\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}$
$u''=-2$	$v_2=\frac{-\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}$
$u'''=0$	$v_3=\frac{\cos\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}$

26. Obtain the half range cosine series for the function $f(x)=x(\pi-x)$ in $0<x<\pi$. Hence deduce that $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+...=\frac{\pi^4}{90}$.

Solution:

Given $f(x)=x(\pi-x)$ in $(0,\pi)$

The Half – range Fourier cosine series is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\,cosnx\ \dots\dots\dots\dots.(1)$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x(\pi-x)\quad dx$$

$$=\frac{2}{\pi}\int_0^{\pi}(x\pi-x^2)\quad dx$$

$$= \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] = \frac{2\pi^3}{\pi}\left[\frac{3-2}{6}\right]$$

$$a_0 = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^\pi x(\pi - x)\cos nx\, dx$$

$u = x\pi - x^2$	$v = \cos nx$
$u' = 2x$	$v_1 = \frac{\sin\ nx}{n}$
$u'' = 2$	$v_2 = \frac{-\cos\ nx}{n^2}$
$u''' = 0$	$v_3 = \frac{-\sin\ nx}{n^3}$

$$= \frac{2}{\pi}\int_0^\pi (x\pi - x^2)\cos nx\, dx$$

using Bernouille's formula $\int uv\, dx = uv_1 - u'v_2 + u''v_3 - \cdots$

$$= \frac{2}{\pi}\left[(x\pi - x^2)\frac{\sin nx}{n} - (\pi - 2x)\left(\frac{-cosnx}{n^2}\right) + (-2)\left(\frac{-sinnx}{n^3}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[(\pi - 2x)\left(\frac{cosnx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[(\pi - 2\pi)\left(\frac{cosn\pi}{n^2}\right) - (\pi - 0)\left(\frac{cos0}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[-\pi\frac{(-1)^n}{n^2}\frac{\pi}{n^2}\right]$$ [$\because \sin 0 = 0 \sin n\pi = 0, \cos 0 = 1\ \&\ \cos n\pi = (-1)^n$]

$$a_n = \frac{-2\pi}{\pi n^2}[(-1)^n + 1] = \frac{-2}{n^2}[(-1)^n + 1]$$

$$\therefore a_n = \begin{cases} 0 & if\ n\ is\ odd \\ \frac{-4}{n^2} & if\ n\ is\ even \end{cases}$$

Substituting a_0 & a_n values in equation (1) we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\, cosnx$$

$$f(x) = \frac{\pi^2}{2*3} - \sum_{n=2,4}^{\infty}\frac{4}{n^2}cosnx$$

$$f(x) = \frac{\pi^2}{6} - 4\sum_{n=2,4}^{\infty}\frac{1}{n^2}cosnx$$

To find the sum of the series, w.k.t the

Parseval's identity for Fourier cosine series is

$$\frac{{a_0}^2}{2} + \sum_{n=`1}^{\infty}{a_n}^2 = \frac{2}{\pi}\int_0^\pi (f(x))^2 dx$$

$$\frac{\left(\frac{\pi^2}{3}\right)^2}{2} + \sum_{n=2,4}^{\infty}\frac{16}{n^4} = \frac{2}{\pi}\int_0^\pi x^2(\pi - x)^2\, dx$$

$$\frac{\left(\frac{\pi^2}{3}\right)^2}{2} + \sum_{n=2,4}^{\infty} \frac{16}{n^4} = \frac{2}{\pi}\int_0^{\pi} (x^2\pi - 2\pi x^3 + x^4)\ dx$$

$$\frac{\pi^4}{18} + \sum_{n=2,4}^{\infty} \frac{16}{n^4} = \frac{2}{\pi}\left[\frac{\pi^2 x^3}{3} - \frac{2\pi x^4}{4} + \frac{x^5}{5}\right]_0^{\pi}$$

$$\frac{\pi^4}{18} + \sum_{n=2,4}^{\infty} \frac{16}{n^4} = \frac{2}{\pi}\left[\frac{\pi^5}{3} - \frac{\pi^5}{2} + \frac{\pi^5}{5}\right]$$

$$\frac{\pi^4}{18} + \sum_{n=2,4}^{\infty} \frac{16}{n^4} = \frac{2\pi^5}{\pi}\left[\frac{10 - 15 + 6}{30}\right]$$

$$\frac{\pi^4}{18} + \sum_{n=2,4}^{\infty} \frac{16}{n^4} = \frac{\pi^4}{15}$$

$$\frac{\pi^4}{18} + \sum_{n=2,4}^{\infty} \frac{16}{n^4} = \frac{\pi^4}{15}$$

$$\sum_{n=2,4}^{\infty} \frac{16}{n^4} = \frac{\pi^4}{15} - \frac{\pi^4}{18}$$

$$\sum_{n=2,4}^{\infty} \frac{16}{n^4} = \frac{18\pi^4 - 15\pi^4}{270}$$

$$\Rightarrow\ 16\sum_{n=1}^{\infty} \frac{1}{(2n)^4} = \frac{3\pi^4}{270}$$

$u = x$	$v = \cos n\pi x/4$
$u' = 1$	$v_1 = \frac{\sin n\pi x/4}{n\pi/4}$
$u'' = 0$	$v_2 = \frac{-\cos n\pi x/4}{n^2\pi^2/16}$

$$\Rightarrow\ \frac{16}{16}\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

27. Obtain the Fourier cosine series expansion of $f(x) = x$ in $0 < x < 4$. Hence deduce the value of $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots + \infty$

Solution:

Given $f(x) = x$

The half range fourier cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{nx\pi}{l} \qquad \ldots\ldots(1)$$

where $\quad a_0 = \frac{2}{l}\int_0^l f(x)dx \quad = \frac{2}{4}\int_0^4 xdx$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^4 = \frac{1}{4}\,[16 - 0]$$

$$a_0 = 4$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{nx\pi}{l}dx$$

$$= \frac{2}{4}\int_0^4 x\cos\frac{nx\pi}{4}dx$$

using Bernouille's formula $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$

$$= 2\left[x\left(\frac{\sin n\pi x/4}{n\pi/4}\right) - 1\left(\frac{-\cos n\pi x/4}{n^2\pi^2/16}\right)\right]_0^4$$

$$= \frac{32}{n^2\pi^2}[(0+(-1)^n) - (0+1]$$

$$= \frac{32}{n^2\pi^2}[(-1)^n - 1]$$

$$a_n = \begin{cases} \frac{64}{n^2\pi^2}, & if\ n\ is\ odd \\ 0, & if\ n\ is\ even \end{cases}$$

Substituting a_0 & a_n value in equation (1) we get

$$f(x) = 2 + \sum_{n=0dd}^{\infty} \frac{64}{n^2\pi^2}\cos\frac{nx\pi}{4}$$

By Parseval's identity $\frac{a_0{}^2}{2} + \sum_{n=1}^{\infty} a_n{}^2 = \frac{2}{l}\int_0^l (f(x))^2 dx$

$$\frac{4}{2} + \frac{256}{\pi^4}\sum_{n=0dd}^{\infty}\frac{1}{n^4} = \frac{2}{4}\int_0^4 x^2 dx$$

$$2 + \frac{256}{\pi^4}\sum_{n=odd}^{\infty}\frac{1}{n^4} = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^4$$

$$= \frac{1}{6}(64-0)$$

$$\frac{256}{\pi^4}\sum_{n=odd}^{\infty}\frac{1}{n^4} = \frac{64}{6}$$

$$\sum_{n=odd}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{96}$$

28. Obtain Compute the first two harmonics of the Fourier series of $f(x)$ given in the following table:

x	**0**	$\frac{T}{6}$	$\frac{T}{3}$	$\frac{T}{2}$	$\frac{2T}{3}$	$\frac{5T}{6}$	T
$f(x)$	**1.98**	**1.30**	**1.05**	**1.30**	**-0.88**	**-0.25**	**1.98**

Solution:

First and last value are same. Hence we omit the last value

When x varies from 0 to T

θ varies from 0 to 2π

We know that the Fourier series is $y = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta \ldots(1)$

x	$\theta = \frac{2\pi x}{T}$	y	$y\cos\theta$	$y\sin\theta$
0	0	1.98	1.98	0
T/6	$\frac{\pi}{3}$	1.30	0.65	1.1258
T/3	$\frac{2\pi}{3}$	1.05	-0.525	0.9093

T/2	π	1.30	-1.3	0
2T/3	$\frac{4\pi}{3}$	-0.88	0.44	0.762
5T/6	$\frac{5\pi}{3}$	-0.25	-0.125	0.2165
Sum		**4.5**	**1.12**	**3.013**

$$a_0 = 2\left[\frac{\sum y}{n}\right] = 2\left[\frac{4.5}{6}\right] = 1.5$$

$$a_1 = 2\left[\frac{\sum y\cos\theta}{n}\right] = 2\left[\frac{1.12}{6}\right] = 0.37\ ;\ b_1 = 2\left[\frac{\sum y\sin\theta}{n}\right] = 2\left[\frac{3.013}{6}\right] = 1.004$$

Substituting the above value in equation (1) we get

$$y = \frac{1.5}{2} + 0.37\cos\theta + 1.004\sin\theta$$

$$y = 0.75 + 0.37\cos\theta + 1.004\sin\theta$$

29. Obtain Compute the first two harmonics of the Fourier series of $f(x)$ given in the following table:

x	0	1	2	3	4	5
y	9	18	24	28	26	20

9 18 24 28 26 20

Soo 9 18 24 28 26 20

We know that the Fourier series is

$$y = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + b_1\sin x + b_2\sin 2x\ \ldots\ldots(1)$$

x	**y**	$\frac{\pi x}{3}$	$\frac{2\pi x}{3}$	$y\cos\frac{\pi x}{3}$	$y\sin\frac{\pi x}{3}$	$y\cos\frac{2\pi x}{3}$	$y\sin\frac{2\pi x}{3}$
0	**9**	0	0	9	0	9	0
1	**18**	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	9	15.7	-9	15.6
2	**24**	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	-12	20.9	-24	0
3	**28**	π	2π	-28	0	28	0
4	**26**	$\frac{4\pi}{3}$	$\frac{8\pi}{3}$	-13	-22.6	-13	22.6
5	**20**	$\frac{10\pi}{3}$	$\frac{10\pi}{3}$	10	-17.4	-10	-17.4
Σ	**125**			**-25**	**-3.4**	**-19**	**20.8**

$$a_0 = 2\left[\frac{\sum y}{n}\right] = 2\left[\frac{125}{6}\right] = 41.66$$

$$a_1 = 2\left[\frac{\sum y\cos\frac{\pi x}{3}}{n}\right] = 2\left[\frac{-25}{6}\right] = -8.33$$

$$a_2 = 2\left[\frac{\sum y\cos\frac{2\pi x}{3}}{n}\right] = 2\left[\frac{-19}{6}\right] = -6.33$$

$$b_1 = 2\left[\frac{\sum y\sin\frac{\pi x}{3}}{n}\right] = 2\left[\frac{-3.4}{6}\right] = -1.13$$

$$b_2 = 2\left[\frac{\sum y\sin\frac{2\pi x}{3}}{n}\right] = 2\left[\frac{20}{6}\right] = 6.9$$

Substituting these values in equation (1) we get

$$y = \frac{41.33}{2} - 8.33cos\frac{\pi x}{3} - 6.33cos\frac{2\pi x}{3} - 1.13sin\frac{\pi x}{3} + 6.9sin\frac{2\pi x}{3}$$

30. Obtain Compute the first two harmonics of the Fourier series of $f(x)$ given in the following table:

x	**0**	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	**2** π
y	**1.0**	**1.4**	**1.9**	**1.7**	**1.5**	**1.2**	**1.0**

Solution:

We know that the Fourier series is

$$y = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + b_1\sin x + b_2\sin 2x \; .....(1)$$

X	y	y cosx	y sin x	y cos 2x	y sin 2x
0	1	1	0	1	0
$\frac{\pi}{3}$	1.4	0.7	1.2124	-0.7	1.2124
$\frac{2\pi}{3}$	1.9	-0.95	1.6454	-0.95	-1.6454
π	1.7	-1.7	0	1.7	0
$\frac{4\pi}{3}$	1.5	-0.75	-1.299	-0.75	1.299
$\frac{5\pi}{3}$	1.2	0.6	-1.039	-0.6	-1.0392
Σ	**8.7**	**-1.1**	**0.5196**	**-0.3**	**-0.1732**

$$a_0 = 2\left[\frac{\sum y}{n}\right] = 2\left[\frac{8.7}{6}\right] = 2.9$$

$$a_1 = 2\left[\frac{\sum y\cos x}{n}\right] = 2\left[\frac{-1.1}{6}\right] = -0.37$$

$$a_2 = 2\left[\frac{\sum y\cos 2x}{n}\right] = 2\left[\frac{-0.3}{6}\right] = -0.1$$

$$b_1 = 2\left[\frac{\sum y\sin x}{n}\right] = 2\left[\frac{0.5196}{6}\right] = 0.17$$

$$b_2 = 2\left[\frac{\sum y \sin 2x}{n}\right] = 2\left[\frac{-0.1732}{6}\right] = -0.06$$

Substituting these values in equation (1) we get

$$y = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x$$

$$y = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + +b_2 \sin 2x)$$

$$y = 1.45 + (-0.37 \cos x + 0.17 \sin x) + (-0.1 \cos 2x - 0.06 \sin 2x)$$

UNIT-III

PART A

1. Write down the possible solution of one dimensional heat equation.

Solution:

The one dimensional heat equation $u_t = \alpha^2 u_{xx}$

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x)Ce^{\alpha^2\lambda^2 t}$$

2. Classify the pde $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$.

Solution:

Given $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

$$u_{xx} - u_t = 0$$

Second order p.d.e in the function 'u' of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) = 0$$

Here $A = 1, B = 0, C = 0$

$\therefore\ B2 - 4AC = 0 - 4(1)(0) = 0$

$\therefore$ The given equation is parabolic equation.

3. Classify the pde $\left(1-x^2\right)z_{xx} - 2xyz_{xy} + \left(1-y^2\right)z_{yy} + xz_x + 3x^2yz_y - 2z = 0$

Solution:

Given $(1 - x^2)z_{xx} - 2xyz_{xy} + (1 - y^2)z_{yy} + xz_x + 3x^2yz_y - 2z = 0$

Here A= $1 - x^2$, B= $-2xy$, $C = 1 - y^2$

$\therefore B^2 - 4AC = 4(x^2 + y^2 - 1)$

The given PDE is elliptic, parabolic and hyperbolic respectively inside, on and outside the circle $x^2 + y^2 = 1$.

4. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y(x,0) = V_0 \sin^3 \frac{\pi x}{l}$. It is released from rest in this position. Write the boundary conditions.

Solution:

The displacement function $y(x,t)$ is the solution of the wave equation.

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are

i) $y(0,t) = 0\ for\ all\ t \geq 0$
ii) $y(l,t) = 0\ for\ all\ t \geq 0$
iii) $\frac{\partial y}{\partial t}(x,0) = 0,\ \ 0 < x < l$
iv) $y(x,0) = f(x) = y_0 \sin\frac{\pi x}{l},\ 0 < x < l.$

5. Write all three possible solutions of steady state two dimensional heat equations.

Solution:

The equation is $\frac{\partial u}{\partial t} = \alpha^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$.

Thus the various possible solution of the heat equation are

i) $y(x,t) = (A_5 \cos \lambda y + A_6 \sin \lambda y)(A_7 e^{\lambda x} + A_8 e^{-\lambda x})$

ii) $y(x,t) = (A_1 \cos \lambda x + A_2 \sin \lambda x)(A_3 e^{\lambda y} + A_4 e^{-\lambda y})$

iii) $u(x,t) = (A_9 x + A_{10})\,(A_{11} y + A_{12}))$

6. Write the three possible solutions of one dimensional heat equation.

Solution:

The one dimensional heat equation $u_t = \alpha^2 u_{xx}$

Thus the various possible solutions are

i) $u(x,t) = (A_1 e^{\lambda x} + B_1 e^{-\lambda x})(C_1 e^{-\alpha^2 \lambda^2 t})$

ii) $u(x,t) = (A_2 \cos \lambda x + B_2 \sin \lambda x) C_2 e^{-\alpha^2 \lambda^2 t}$

iii) $u(x,t) = (A_3 x + B_3) C_3$

7. A rod 40cm long with insulated sides has its ends A and B kept at 20˚C and 60˚C respectively. Find the steady state temperature at a location 15cm from A.

Solution:

W.K.T $u(x) = ax + b, 0 < x < 40$

If $x = 0 \Rightarrow u(0) = b \Rightarrow 20 = b$ and if $x = 40 \Rightarrow u(40) = 40a + 20 \Rightarrow 60 = 40a + 20$

$\therefore a = 1 \quad \Rightarrow \quad u(x) \quad = x + 20$

At $x = 15$ => $u(x) = 15 + 20 = 35$.

8. Write down the three possible solutions of Laplace equation in two dimensions.

Solution:

The Laplace equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Thus the various possible solution of the heat equation are

iv) $y(x,t) = (A_5 \cos \lambda y + A_6 \sin \lambda y)(A_7 e^{\lambda x} + A_8 e^{-\lambda x})$

v) $y(x,t) = (A_1 \cos \lambda x + A_2 \sin \lambda x)(A_3 e^{\lambda y} + A_4 e^{-\lambda y})$

vi) $u(x,t) = (A_9 x + A_{10})\,(A_{11} y + A_{12}))$

9. A plate is bounded by the lines $x = 0, y = 0, x = l \;\&\; y = l$. It faces are insulated. The edge coinciding with x axis is kept at 100˚.The edge coinciding with y axis is kept at 50˚. The other two edges are kept at 0˚C. Write the boundary condition needed for solving the two dimensional heat flow.

Solution:

(i) $u(x,0) = 100^{\circ}c, for\ o < x < l$

(ii) $u(0,y) = 50^{\circ}c\ , for\ o < x < l$

(iii) $u(x,l) = 0^{\circ}c\ , for\ 0 < x < l$

(iv) $u(l,y) = 0^{\circ}c\ , for\ 0 < y < l$

10. An insulated rod of length 60cm has its ends A and B maintained at 20˚C and 80˚C respectively. Find the steady state solution of the rod.

Solution:

The steady state equation of one dimensional heat flow is

$$\frac{d^2u}{dx^2} = 0 \quad \ldots.(1)$$

The general solution of (1) is $u(x) = ax + b \quad \ldots\ldots.(2)$

The boundary conditions are $u(0) = 20, and\ u(60) = 80$

Put $x = 0$ in equ(2), $u(0) = 0 + b = 20$

$$b = 20$$

Put $x = 60$ in equ(2), $u(60) = 60a + 20 = 80$

$$60a + 20 = 80$$

$$60a = 80 - 20 = 60$$

$$a = \frac{60}{60} = 1$$

$$a = 1$$

Sub a & b value in eqn (2), we get

$$u(x) = x + 20$$

11. In the one dimensional heat equation $u_t = c^2 u_{xx,}$ what is c^2?

Solution:

$$a^2 = \frac{T}{m} = \frac{\text{Tension}}{\text{mass per unit length of the string}}$$

12. What is the basic difference between the solution of one dimensional wave equation and one dimensional heat equation with respect to time?

Solution:

One dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$	One dimensional heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$
It is classified as hyperbolic pde	It is classified as parabolic pde
Suitable solution of one dimensional wave equation is $y(x,t) = (A_2 \cos \lambda x + B_2 \sin \lambda x)(C_2 \cos \lambda at + D_2 \sin \lambda at)$	Suitable solution of one dimensional heat equation is $u(x,t) = (A_2 \cos \lambda x + B_2 \sin \lambda x) C_2 e^{\alpha^2 \lambda^2 t}$

13. Write down the one-dimensional heat equation both in transient and steady states.
Solution:

The two dimensional heat equation in transient state is

$$\frac{\partial u}{\partial t} = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

The two dimensional heat equation in transient state is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

14. Define steady state condition on heat flow.
Solution:

Steady state condition in heat flow means that the temperature at any point in the body does not vary with time.

ie., it is independent of t, the time

In steady state one dimensional heat equation is $\frac{d^2u}{dx^2} = 0$

15. State the assumptions in deriving the one dimensions heat flow equation.

Solution:

i) Heat flows from higher to lower temp.

ii) The rate at which heat flows across any area is proportional to the area and to the temp. Gradient normal to the curve. This constant of proportionality is known as the thermal conductivity (k) of the material. It is known as Fourier law of heat conduction.

PART-B

16. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y(x,0) = k(lx - x^2), 0 < x < l$. It is released from rest from this position. Determine the expression for the displacement at any time t.

Solution:

We know that the one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

The boundary conditions are,

(i) $y(0,t) = 0$ for all $t \geq 0$

(ii) $y(l,t) = 0$ for all $t \geq 0$

(iii) $\frac{\partial y}{\partial t}(x,0) = 0$ (Since the initial velocity is zero)

(iv) $y(x,0) = k(lx - x^2)$

The suitable solution which satisfies our boundary conditions is

$$\boldsymbol{y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)}\text{.........}(1)$$

Apply condition (i) in (1) we get

$$y(0,t) = 0$$

$$(A\cos 0 + B\sin 0)(C\cos\lambda at + D\sin\lambda at) = 0$$

$$(A + 0)(C\cos\lambda at + D\sin\lambda at) = 0$$

$$\Rightarrow \boldsymbol{A = 0} \qquad [\because (C\cos\lambda at + D\sin\lambda at) \neq 0]$$

Substitute $A = 0$ in (1) we get

$$y(x,t) = (B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at)\text{..........}(2)$$

Apply condition (ii) in (2) we get $y(l,t) = 0$

$$(B\sin\lambda l)(C\cos\lambda at + D\sin\lambda at) = 0$$

$$B\sin\lambda l = 0 \qquad [\because (c\cos\lambda at + D\sin\lambda at) \neq 0]$$

If $B = 0$ we get a trivial solution.

$\therefore B \neq 0$ and $\sin\lambda l = 0$

$$sin\lambda l = sinn\pi$$

$$\lambda l = n\pi$$

$$\boldsymbol{\lambda = \frac{n\pi}{l}}$$

Substitute $\lambda = \frac{n\pi}{l}$ in (2) we get

$$y(x,t) = (B \sin\frac{n\pi}{l}x)(C \cos\frac{n\pi}{l}at + D\ sin\frac{n\pi}{l}at)..........(3)$$

To Apply condition (iii) differentiate (3) partially w.r.t 't' we get

$$\frac{\partial y}{\partial t}(x,t) = \left(B \sin\frac{n\pi}{l}x\right)\left(-C\left(\frac{n\pi a}{l}\right) sin\frac{n\pi}{l}at + D\left(\frac{n\pi a}{l}\right) cos\frac{n\pi}{l}at\right)$$

Now Apply condition (iii) we get

$$\frac{\partial y}{\partial t}(x,0) = 0$$

$$\left(B \sin\frac{n\pi}{l}x\right)\left(D\left(\frac{n\pi a}{l}\right) cos\frac{n\pi}{l}at\right) = 0$$

Since B ≠0 $\therefore\ \boldsymbol{D = 0}$

Substitute $D = 0$ in (3) we get

$$y(x,t) = (B \sin\frac{n\pi}{l}x)(c \cos\frac{n\pi}{l}at)$$

$$= BC\ \sin\frac{n\pi}{l}x \cos\frac{n\pi}{l}at$$

$$y(x,t) = B_n \sin\frac{n\pi}{l}x \cos\frac{n\pi}{l}at \quad \text{where } B_n = BC$$

The most general solution is $\boldsymbol{y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x \cos\frac{n\pi}{l}at}$.....................(4)

Applying the boundary conditions (iv) in (4) we get

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x = k(lx - x^2)$$

To find B_n expand $k(lx - x^2)$ in half range Fourier sine series in the interval $(0, l)$

$u = (lx - x^2)$	$v = \sin\frac{n\pi x}{l}$
$u' = (l - 2x)$	$v_1 = -\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}$
$u'' = (-2)$	$v_2 = -\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}$
$u''' = 0$	$v_3 = +\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}$

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\ dx$$

$$= \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{n\pi x}{l}\ dx$$

Using Bernoulli formula $\int uvdx = uv_1 - u'v_2 + u''v_3 - \ldots$

$$= \frac{2k}{l}\left[(lx - x^2)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l - 2x)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right]_0^l$$

$$= \frac{2k}{l}\left[-(lx - x^2)\frac{l}{n\pi}\cos\frac{n\pi x}{l} + (l - 2x)\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} - 2\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2k}{l}\left[0 + 0 - \frac{2l^3}{n^3\pi^3}\cos\frac{n\pi l}{l} - 0 - 0 + \frac{2l^3}{n^3\pi^3}\cos 0\right]$$

$$= \frac{2k}{l}\left[-\frac{2l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{2k}{l}\left[\frac{2l^3}{n^3\pi^3}(1 - (-1)^n)\right]$$

$$= \frac{4k}{l}\left[\frac{l^3}{n^3\pi^3}(1 - (-1)^n)\right]$$

$$B_n = \begin{cases} 0, if\ n\ is\ even \\ \frac{8kl^2}{n^3\pi^3}, if\ n\ is\ odd \end{cases}$$

Substitute the value of B's in (4) we get

$$y(x,t) = \sum_{odd}\frac{8kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

17. A tightly stretched string with fixed end points $x = 0$ *and* $x = l$ is initially at rest in equilibrium position. If it is set vibrating giving each point a velocity $\lambda x(l - x)$, determine the displacement of any point on the string at a distance x from one end at any time t.

Solution:

We know that the one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

The boundary conditions are,

(i) $y(0,t) = 0$ for all $t \geq 0$

(ii) $y(l,t) = 0$ for all $t \geq 0$,

(iii) $y(x,0) = 0$ for $0 \leq x \leq l$

$$(iv)\frac{\partial y}{\partial t}(x,0) = \lambda x(l-x), for\ 0 \le x \le l$$

The suitable solution which satisfies our boundary conditions is

$$\boldsymbol{y(x,t) = (A\cos\lambda x + B\sin\lambda x)\ (C\cos\lambda at + D\ sin\lambda at)}..........(1)$$

Apply condition (i) in (1) we get

$$y(0,t) = 0$$

$$(A\cos 0 + B\sin 0)\ (C\cos\lambda at + D\ sin\lambda at) = 0$$

$$(A+0)(C\cos\lambda at + D\ sin\lambda at) = 0$$

$$\Rightarrow \boldsymbol{A = 0} \qquad [\because (C\cos\lambda at + D\ sin\lambda at) \ne 0]$$

Substitute $A = 0$in (1) we get

$$y(x,t) = (B\sin\lambda x)(C\cos\lambda at + D\ sin\lambda at)...........(2)$$

Apply condition (ii) in (2) we get $y(l,t) = 0$

$$(B\sin\lambda l)(C\cos\lambda at + D\ sin\lambda at) = 0$$

$$Bsin\lambda l = 0 \qquad [\because (c\cos\lambda at + D\ sin\lambda at) \ne 0]$$

If $B = 0$ we get a trivial solution.

∴B ≠0 and $sin\lambda l = 0$

$$sin\lambda l = sinn\pi$$

$$\lambda l = n\pi$$

$$\boldsymbol{\lambda = \frac{n\pi}{l}}$$

Substitute $\lambda = \frac{n\pi}{l}$ in (2) we get

$$y(x,t) = (B\sin\frac{n\pi}{l}x)(C\cos\frac{n\pi}{l}at + D\ sin\frac{n\pi}{l}at)..........(3)$$

Apply condition (iii) in (3) we get $y(x,0) = 0$

$$(B\sin\frac{n\pi}{l}x)(C\cos 0 + D\ sin 0) = B\sin\frac{n\pi x}{l}\ C = 0$$

Here $\sin\frac{n\pi}{l}x \ne 0$, $\quad\therefore \quad \boldsymbol{C = 0}$

Substitute $C = 0$ in (3) we get

$$y(x,t) = (B\sin\frac{n\pi}{l}x)(D\sin\frac{n\pi}{l}at)$$

$$= BD\ \sin\frac{n\pi}{l}x\sin\frac{n\pi}{l}at$$

$$y(x,t) = B_n\sin\frac{n\pi}{l}x\sin\frac{n\pi}{l}at \qquad \text{where } B_n = BD$$

The most general solution is

$$\boldsymbol{y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi}{l}x\sin\frac{n\pi}{l}at}......................\textbf{(4)}$$

To Apply condition (iv) differentiate (4) partially w.r.t 't' we get

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} B_n\frac{n\pi a}{l}\ \sin\frac{n\pi}{l}x\cos\frac{n\pi}{l}at \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Apply condition (iv) in (5) we get

$$\frac{\partial y}{\partial t}(x,0)=\sum_{n=1}^{\infty} B_n \frac{n\pi a}{l}\ \sin\frac{n\pi}{l}x=\lambda x(l-x)$$

To find B_n expand in half range Fourier sine series in the interval $(0,l)$

$$B_n\frac{n\pi a}{l}=\frac{2}{l}\int_0^l \lambda x(l-x)\sin\frac{n\pi}{l}x\ dx$$

$$=\frac{2\lambda}{l}\int_0^l (lx-x^2)\sin\frac{n\pi}{l}x\ dx$$

Using Bernoulli formula $\int uv dx=uv_1-u'v_2+u''v_3-\ \ldots$

$u=(lx-x^2)$	$v=\sin\frac{n\pi x}{l}$
$u'=(l-2x)$	$v_1=-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}$
$u''=(-2)$	$v_2=-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}$
$u'''=0$	$v_3=+\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}$

$$=\frac{2\lambda}{l}\left[-(lx-x^2)\frac{\cos\frac{n\pi}{l}x}{\frac{n\pi}{l}}+(l-2x)\frac{\sin\frac{n\pi}{l}x}{\frac{n^2\pi^2}{l^2}}-2\frac{\cos\frac{n\pi}{l}x}{\frac{n^3\pi^3}{l^3}}\right]_0^l$$

$$=\frac{2\lambda}{l}\left[0+0-\frac{2l^3}{n^3\pi^3}\cos n\pi+0-0+\frac{2l^3}{n^3\pi^3}\right]$$

$$=\frac{4\lambda l^2}{n^3\pi^3}[-\cos n\pi+1]$$

$$=\frac{4\lambda l^2}{n^3\pi^3}[1-(-1)^n]$$

$$=\begin{cases}0, if\ n\ is even\\ \frac{8\lambda l^2}{n^3\pi^3}, if\ n\ is\ odd\end{cases}$$

$$B_n=\begin{cases}0, if\ n\ is even\\ \frac{8\lambda l^3}{an^4\pi^4}, if\ n\ is\ odd\end{cases}$$

Now substitute the value of B_n in (5) we get

$$y(x,t)=\sum_{n=odd}^{\infty}\frac{8\lambda l^3}{an^4\pi^4}\sin\frac{n\pi}{l}x\sin\frac{n\pi}{l}at$$

$$y(x,t)=\frac{8\lambda l^3}{a\pi^4}\sum_{n=odd}^{\infty}\frac{1}{n^4}\sin\frac{n\pi}{l}x\sin\frac{n\pi}{l}at$$

18. A **tightly stretched string of length l has its ends fastened at** $x=0$ *and* $x=l$. **The midpoint of the string is taken to a height b and released from rest in that position. Determine the displacement of a point of the string at time t from the instant release.**

Solution:

We know that the one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2}=a^2\ \frac{\partial^2 y}{\partial x^2}$

The boundary conditions are,

(i) $y(0,t) = 0$ for all $t \geq 0$

(ii) $y(2l,t) = 0$ for all $t \geq 0$,

(iii) $\frac{\partial y}{\partial t}(x,0) = 0,\ 0 \leq x \leq 2l$

The equation of OA is $O(0,0)$ & $A(l,b)$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-0}{b-0} = \frac{x-0}{l-0}$$

$$\frac{y}{b} = \frac{x}{l} \Rightarrow y = \frac{bx}{l}$$

The equation of BA is $B(2l,0)$ & $A(l,b)$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-0}{b-0} = \frac{x-2l}{l-2l}$$

$$\frac{y}{b} = \frac{x-2l}{-l} \quad \Rightarrow \quad y = \frac{-b}{l}(x-2l), \qquad l \leq x \leq 2l$$

$$(iv)\ y(x,0) = \begin{cases} \frac{bx}{l}, & 0 \leq x \leq l \\ \frac{-b}{l}(x-2l), & l \leq x \leq 2l \end{cases}$$

The suitable solution which satisfies our boundary conditions is

$$\boldsymbol{y(x,t) = (A\cos\lambda x + B\sin\lambda x)\,(C\cos\lambda at + D\,sin\lambda at)}\ldots\ldots\ldots(1)$$

Apply condition (i) in (1) we get

$$y(0,t) = 0$$

$$(A\cos 0 + B\sin 0)\,(C\cos\lambda at + D\,sin\lambda at) = 0$$

$$(A+0)(C\cos\lambda at + D\,sin\lambda at) = 0$$

$$\Rightarrow \boldsymbol{A = 0} \qquad [\because (C\cos\lambda at + D\,sin\lambda at) \neq 0]$$

Substitute $A = 0$in (1) we get

$$y(x,t) = (B\sin\lambda x)(C\cos\lambda at + D\,sin\lambda at)\ldots\ldots\ldots.(2)$$

Apply condition (ii) in (2) we get $y(2l,t) = 0$

$$(B\sin\lambda 2l)(C\cos\lambda at + D\,sin\lambda at) = 0$$

$$Bsin2\lambda l = 0 \qquad [\because (c\cos\lambda at + D\,sin\lambda at) \neq 0]$$

If $B = 0$ we get a trivial solution.

∴B ≠0 and $sin2\lambda l = 0$

$$sin2\lambda l = sinn\pi$$

$$2\lambda l = n\pi$$

$$\boldsymbol{\lambda = \frac{n\pi}{2l}}$$

Substitute $\lambda = \frac{n\pi}{2l}$ in (2) we get

$$y(x,t) = (B\sin\frac{n\pi}{2l}x)(C\cos\frac{n\pi}{2l}at + D\,sin\frac{n\pi}{2l}at)\dots\dots\dots(3)$$

To Apply condition (iii) differentiate (3) partially w.k.t 't' we get

$$\frac{\partial y}{\partial t}(x,t) = \left(B\sin\frac{n\pi}{2l}x\right)\left(-C\left(\frac{n\pi a}{2l}\right)sin\frac{n\pi}{2l}at + D\left(\frac{n\pi a}{2l}\right)cos\frac{n\pi}{2l}at\right)$$

Now Apply condition (iii) we get $\frac{\partial y}{\partial t}(x,0) = 0$

$$\left(B\sin\frac{n\pi}{2l}x\right)\left(D\left(\frac{n\pi a}{2l}\right)cos\frac{n\pi}{2l}at\right) = 0$$

Since B ≠0 $\therefore \boldsymbol{D = 0}$

Substitute $D = 0$ in (3) we get

$$y(x,t) = (B\sin\frac{n\pi}{2l}x)(c\cos\frac{n\pi}{2l}at)$$

$$= BC\ \sin\frac{n\pi}{2l}x\cos\frac{n\pi}{2l}at$$

$y(x,t) = B_n\sin\frac{n\pi}{2l}x\cos\frac{n\pi}{2l}at$ where $B_n = BC$

The most general solution is

$$\boldsymbol{y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi}{2l}x\cos\frac{n\pi}{2l}at}\dots\dots\dots\dots\dots\dots\dots(4)$$

Applying the boundary conditions (iv) in (4) we get

$$y(x,0) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi}{l}x$$

$$= \begin{cases} \frac{bx}{l}, 0 \le x \le l \\ \frac{-b}{l}(x-2l), \quad l \le x \le 2l \end{cases} \dots\dots\dots\dots\dots.(5)$$

To find B_n expand in half range Fourier sine series in the interval (0,l)

$$B_n = \frac{2}{2l}\int_0^{2l} f(x)\sin\frac{n\pi}{2l}x\ dx$$

$u = x$	$u = x - 2l$	$v = \sin\frac{n\pi x}{2l}$
$u' = 1$	$u' = 1$	$v_1 = -\frac{\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}$
$u'' = 0$	$u'' = 0$	$v_2 = -\frac{\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}$

$$B_n = \frac{1}{l}\int_0^l \frac{bx}{l}\sin\frac{n\pi}{2l}x\ dx - \frac{1}{l}\int_l^{2l}\frac{b}{l}(x-2l)\sin\frac{n\pi}{2l}x\ dx$$

$$= \frac{b}{l^2}\int_0^l x\sin\frac{n\pi}{2l}x\ dx - \frac{b}{l^2}\int_l^{2l}(x-2l)\sin\frac{n\pi}{2l}x\ dx$$

Using Bernoulli formula $\int uvdx = uv_1 - u'v_2 + u''v_3 - \dots$

$$= \frac{b}{l^2}\left[(x)\left(-\frac{\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - (1)\left(-\frac{\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}\right)\right]_0^l$$

$$-\frac{b}{l^2}\left[(x-2l)\left(-\frac{\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - (1)\left(-\frac{\sin\frac{n\pi x}{2l}}{\frac{n^2\pi^2}{4l^2}}\right)\right]_l^{2l}$$

$$= \frac{b}{l^2}\left[\frac{-2xl}{n\pi}\cos\frac{n\pi x}{2l} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi x}{2l}\right]_0^l$$

$$-\frac{b}{l^2}\left[\frac{-2l(x-2l)}{n\pi}\cos\frac{n\pi x}{2l} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi x}{2l}\right]_l^{2l}$$

$$= \frac{b}{l^2}\left[\frac{-2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + 0 - 0\right]$$

$$+\frac{b}{l^2}\left[-0 - 0 + \frac{2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{b}{l^2}\left[\frac{-2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{b}{l^2}\left[\frac{8l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$B_n = \begin{cases} 0, if\ n\ is\ even \\ \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}, if\ n\ is\ odd \end{cases}$$

Now substitute the value of B_n in (5) we get

$$y(x,t) = \sum_{n=odd}^{\infty} \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi}{2l}x\cos\frac{n\pi}{2l}at$$

$$= \frac{8b}{\pi^2}\sum_{n=odd}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi}{2l}x\cos\frac{n\pi}{2l}at$$

$$= \frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin\frac{(2n-1)\pi}{2}\sin\frac{(2n-1)\pi}{2l}x\cos\frac{(2n-1)\pi}{2l}at$$

$$y(x,t) = \frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\sin\frac{(2n-1)\pi}{2l}x\cos\frac{(2n-1)\pi}{2l}at$$

19. A tightly stretched string of length l is initially at rest in its equilibrium position and each of its points is given the velocity $V_0 \sin^3 \frac{\pi x}{l}$. Determine the transverse displacement $y(x,t)$.

Solution:

We know that the one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

The boundary conditions are,

(i) $y(0,t) = 0$ for all $t \geq 0$

(ii) $y(l,t) = 0$ for all $t \geq 0$,

(iii) $y(x,0) = 0\ for\ 0 \leq x \leq l$

$(iv) \frac{\partial y}{\partial t}(x,0) = v_0 sin^3(\frac{\pi x}{l})$, $for\ 0 \le x \le l$

The suitable solution which satisfies our boundary conditions is

$$\boldsymbol{y(x,t) = (A\cos\lambda x + B\sin\lambda x)\ (C\cos\lambda at + D\ sin\lambda at)}\ \textbf{..........(1)}$$

Apply condition (i) in (1) we get $y(0,t) = 0$

$$(A\cos 0 + B\sin 0)\ (C\cos\lambda at + D\ sin\lambda at) = 0$$

$$(A + 0)(C\cos\lambda at + D\ sin\lambda at)$$

$$\Rightarrow \boldsymbol{A = 0} \qquad [\because (C\cos\lambda at + D\ sin\lambda at) \neq 0]$$

Substitute $A = 0$in (1) we get

$$y(x,t) = (B\sin\lambda x)(C\cos\lambda at + D\ sin\lambda at)\text{...........(2)}$$

Apply condition (ii) in (2) we get $y(l,t) = 0$

$$(B\sin\lambda l)(C\cos\lambda at + D\ sin\lambda at) = 0$$

$$Bsin\lambda l = 0 \qquad [\because (c\cos\lambda at + D\ sin\lambda at) \neq 0]$$

If $B = 0$ we get a trivial solution.

∴B ≠0 and $sin\lambda l = 0$

$$sin\lambda l = sinn\pi$$

$$\lambda l = n\pi$$

$$\boldsymbol{\lambda = \frac{n\pi}{l}}$$

Substitute $\lambda = \frac{n\pi}{l}$ in (2) we get

$$y(x,t) = (B\sin\frac{n\pi}{l}x)(C\cos\frac{n\pi}{l}at + D\ sin\frac{n\pi}{l}at)\text{..........(3)}$$

Apply condition (iii) in (3) we get

$$y(x,0) = 0$$

$$(B\sin\frac{n\pi}{l}x)(C\cos 0 + D\ sin 0) = 0$$

$$B\sin\left(\frac{n\pi}{l}x\right)\ C = 0$$

Here $\sin\frac{n\pi}{l}x \neq 0$, $\therefore \boldsymbol{C = 0}$

Substitute $C = 0$ in (3) we get

$$y(x,t) = (B\sin\frac{n\pi}{l}x)(D\sin\frac{n\pi}{l}at)$$

$$= BD\ \sin\frac{n\pi}{l}x\sin\frac{n\pi}{l}at$$

$$y(x,t) = B_n\sin\frac{n\pi}{l}x\sin\frac{n\pi}{l}at \qquad \text{where } B_n = BD$$

The most general solution is

$$\boldsymbol{y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x\sin\frac{n\pi}{l}at}\text{......................(4)}$$

To Apply condition (iv) differentiate (4) partially w.r.t ‘t’ we get

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \sin\frac{n\pi}{l}x \cos\frac{n\pi}{l}at \dots\dots\dots\dots\dots\dots\dots (5)$$

Apply condition (iv) in (5) we get

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l} \sin\frac{n\pi}{l}x = v_0 sin^3\left(\frac{\pi x}{l}\right)$$

$$B_1\frac{\pi a}{l}sin\frac{\pi x}{l} + B_2\frac{2\pi a}{l}sin\frac{2\pi x}{l} + B_3\frac{3\pi}{l}sin\frac{3\pi x}{l} + \cdots = 3\frac{v_0}{4}sin\frac{\pi x}{l} - \frac{v_0}{4}sin\frac{3\pi x}{l}$$

[Since $sin^3x = \frac{3sinx - sin3}{4}$]

Now Equating the like terms we get

$$\frac{\pi a}{l}B_1 = 3\frac{v_0}{4}, \frac{2\pi}{l}B_2 = 0, \frac{3\pi a}{l}B_3 = -\frac{v_0}{4}, \frac{4\pi a}{l}B_4 = 0, B_5 = 0, \dots$$

$$B_1 = 3\frac{v_0 l}{4\pi a},\ B_2 = 0,\ B_3 = -\frac{v_0 l}{12\pi a},\ B_4 = 0, B_5 = 0$$

Substitute the value of B.C's in (5) we get

$$y(x,t) = \sum_{n=1}^{\infty} B_n sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$= B_1 sin\frac{\pi x}{l}\sin\frac{\pi at}{l} + B_2 sin\frac{2\pi x}{l}\sin\frac{2\pi at}{l} + B_3 sin\frac{3\pi x}{l}\sin\frac{3\pi at}{l} + \cdots$$

$$y(x,t) = 3\frac{v_0 l}{4\pi a}\ sin\frac{\pi x}{l}\ \sin\frac{\pi at}{l} - \frac{v_0 l}{12\pi a}sin\frac{3\pi}{l}\sin\frac{3\pi at}{l}$$

20. An infinitely long rectangular plate with insulated surfaces is 10cm wide. The two long edges and one short edge are kept at 0˚C,while the other short edge $x = 0$ is kept at temperature $u = \begin{cases} 20y, & 0 \le y \le 5 \\ 20(10-y), & 5 \le y \le 10. \end{cases}$ Construct the steady state temperature distribution in the plate.

Solution:

The two dimensional heat equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = 0$

The boundary Conditions are

(i) $u(x,0) = 0$ *for all x*

(ii) $u(x,10) = 0$ *for all x*

(iii) $u(\infty, y) = 0, 0 \le y \le 10$

(iv) $u(0,y) = \begin{cases} 20y, & 0 \le y \le 5 \\ 20(10-y), & 5 \le y \le 10 \end{cases}$

Now the suitable solution which satisfies our boundary conditions is

$$\boldsymbol{u(x,y) = (A\cos\lambda y + B\sin\lambda y)(C\,e^{\lambda x} + De^{-\lambda x})} \dots\dots\dots\dots (\mathbf{1})$$

Apply condition (i) *in* (1) we get

$$u(x,0) = (A\cos 0 + B\sin 0)(Ce^{\lambda x} + De^{-\lambda x}) = 0$$

$$A(Ce^{\lambda x} + De^{-\lambda x}) = 0$$

$$Ce^{\lambda x} + De^{-\lambda x} \ne 0, \qquad \therefore \boldsymbol{A = 0}$$

Substitute $A =$ 0 in (1) we get

$$u(x,y) = (B\sin\lambda y)\left(C\,e^{\lambda x} + De^{-\lambda x}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Apply condition (ii) in (2) we get

$$u\,(x\,,10) \;= (B\sin\lambda 10)\left(C\,e^{\lambda x} + De^{-\lambda x}\right) = 0$$

$$C\,e^{\lambda x} + De^{-\lambda x} \neq 0, \qquad \therefore \sin\lambda 10 = 0$$

$$\sin\lambda 10 = \sin n\pi$$

$$\lambda 10 = n\pi \Rightarrow \boldsymbol{\lambda} = \frac{\boldsymbol{n\pi}}{\mathbf{10}}$$

Substitute $\lambda = \frac{n\pi}{10}$ in (2) we get

$$u(x,y) = \left(B\sin\frac{n\pi}{10}y\right)\left(C\,e^{\frac{n\pi}{10}x} + De^{-\frac{n\pi}{10}x}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Apply condition (iii) in (3) we get

$$u(\infty,y) = \left(B\sin\frac{n\pi}{10}y\right)\left(C\,e^{\infty} + De^{-\infty}\right) = 0$$

$$B\sin\frac{n\pi}{10}y \neq 0,\; D \neq 0 \qquad \therefore \boldsymbol{C = 0}$$

Substitute $C = 0$ in (3) we get $\quad u(x,y) = \left(B\sin\frac{n\pi}{10}y\right)\left(De^{-\frac{n\pi}{10}x}\right)$

$$= B\,\mathrm{D}\sin\frac{n\pi}{10}y\;\; e^{-\frac{n\pi}{10}x}$$

The most general solution is

$$\boldsymbol{u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{10}y\;\; e^{-\frac{n\pi}{10}x}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Apply condition (iv) in (5) we get

$$u(0,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{10}y \;\; = \begin{cases} 20y, & 0 \le y \le 5 \\ 20(10-y), & 5 \le y \le 10 \end{cases}$$

To find B_n expand in a half range Fourier sine series in [0,10]

$$B_n = \frac{2}{10}\left[\int_0^5 20\,y\,\sin\frac{n\pi y}{10}\,dy + \int_5^{10} 20(10-y)\,\sin\frac{n\pi y}{10}\,dy\right]$$

$$= \frac{40}{10}\left[\int_0^5 y\,\sin\frac{n\pi y}{10}\,dy + \int_5^{10} (10-y)\,\sin\frac{n\pi y}{10}\,dy\right]$$

Using Bernoulli formula $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \ldots$

$u = y$	$u = 10 - y$	$v = \sin\frac{n\pi}{10}y$
$u' = 1$	$u' = -1$	$v_1 = -\frac{\cos\frac{n\pi y}{10}}{\frac{n\pi}{10}}$
$u'' = 0$	$u'' = 0$	$v_2 = -\frac{\sin\frac{n\pi y}{10}}{\frac{n^2\pi^2}{10^2}}$

$$= 4\left\{\left[(y)\left(-\frac{\cos\frac{n\pi y}{10}}{\frac{n\pi}{10}}\right) - (1)\left(-\frac{\sin\frac{n\pi y}{10}}{\frac{n^2\pi^2}{100}}\right)\right]_0^5 + \left[(10-y)\left(-\frac{\cos\frac{n\pi y}{10}}{\frac{n\pi}{10}}\right) - (-1)\left(-\frac{\sin\frac{n\pi y}{10}}{\frac{n^2\pi^2}{100}}\right)\right]_5^{10}\right\}$$

$$= 4\left\{\left[-y\,\frac{\cos\frac{n\pi y}{10}}{\frac{n\pi}{10}} + \frac{\sin\frac{n\pi y}{10}}{\frac{n^2\pi^2}{100}}\right]_0^5 + \left[-(10-y)\,\frac{\cos\frac{n\pi y}{10}}{\frac{n\pi}{10}} - \frac{\sin\frac{n\pi y}{10}}{\frac{n^2\pi^2}{100}}\right]_5^{10}\right\}$$

$$= 4\left[\frac{-50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2} + 0 - 0 - 0 - 0 + \frac{50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$B_n = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Now substitute the value of B_n in (4) we get

$$u(x,y) = \sum_{n=1}^{\infty}\frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi}{10}y\,e^{-\frac{n\pi}{10}x}$$

21. A square plate is bounded by the lines $x=0, x=a, y=0 \,\&\, y=b$. Its surfaces are insulated. The temperature along $y=b$ is kept at 100˚C while the other three edges are kept at 0˚C. Determine the steady state temperature distribution in the plate.

Solution:

The two dimensional heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

The boundary Conditions are

(i) $u(0,y) = 0, \quad 0 \le y \le b$

$(ii$ $u(a,y) = 0, \quad 0 \le y \le b$

$(iii) u(x,0) = 0, 0 \le x \le a,$

(iv) $u(x,b) = 100, \; 0 \le x \le a$

By the method of separation of variables, best solution is

$$\boldsymbol{u(x,y) = (A\cos\lambda x + B\sin\lambda x)(C\,e^{\lambda y} + D\,e^{-\lambda y}) \ldots\ldots\ldots\ldots\ldots(1)}$$

Apply condition (i) in (1) we get

$$u(0,y) = (A\cos 0 + B\sin 0)(C\,e^{\lambda y} + D\,e^{-\lambda y}) = 0$$

$$A\left(C\,e^{\lambda y}+De^{-\lambda y}\right)=0$$

$$\therefore \boldsymbol{A=0}$$

Substitute $A=0$ in (1)

$$u(x,y)=\left(B\sin\lambda x\right)\left(C\,e^{\lambda y}+D\,e^{-\lambda}\ \right)\ldots\ldots\ldots\ldots\ldots(2)$$

Apply condition (ii) in (2) we get

$$u\,(a\,,y)\;=\;\left(B\sin\lambda a\right)\left(C\,e^{\lambda y}+D\,e^{-\lambda y}\right)=0$$

$$B\neq 0,\quad \therefore \sin\lambda a=0=\sin n\pi$$

$$\lambda a=n\pi$$

$$\boldsymbol{\lambda=\frac{n\pi}{a}}$$

Substitute $\lambda=\frac{n\pi}{a}$ in (2)

$$u(x,y)=\left(B\sin\frac{n\pi}{a}x\right)\left(C\,e^{\frac{n\pi}{a}y}+D\,e^{-\frac{n\pi}{a}y}\right)\ldots\ldots\ldots\ldots\ldots(3)$$

Apply condition (iii) in (3) we get

$$u\,(\,x,0)\;=\left(B\sin\frac{n\pi}{a}x\right)\left(C\,e^{\frac{n\pi}{a}(0)}+D\,e^{-\frac{n\pi}{a}(0)}\right)=0$$

$$C\;+D=0\qquad\qquad [\because B\neq 0]$$

$$C\;=-D\Rightarrow\quad \boldsymbol{D=-C}$$

Substitute $D=-C$ in (3)

$$u(x,y)=\left(B\sin\frac{n\pi}{a}x\right)\left(C\,e^{\frac{n\pi}{a}y}-C\,e^{-\frac{n\pi}{a}y}\right)$$

$$=BC\sin\frac{n\pi}{a}x\left[e^{\frac{n\pi}{a}y}-e^{-\frac{n\pi}{a}y}\right]$$

$$=2BC\sin\frac{n\pi}{a}x\left[\frac{e^{\frac{n\pi}{a}y}-e^{-\frac{n\pi}{a}y}}{2}\right]\qquad [multiply\ \&\ divide\ by\ 2)$$

$$=2BC\sin\frac{n\pi}{a}x\left[\sinh\frac{n\pi}{a}y\right]\qquad [\text{since } \sinh\frac{n\pi}{a}y=\frac{e^{\frac{n\pi}{a}y}-e^{-\frac{n\pi}{a}y}}{2}]$$

The most general solution is

$$\boldsymbol{u(x,y)=\sum_{n=\square}^{\infty}\mathrm{B_n}\sin\frac{n\pi}{a}x}\left[\sinh\frac{n\pi}{a}y\right]\ldots\ldots\ldots\ldots(\mathbf{4})\qquad [\because \mathrm{B_n}=2\mathrm{BC}]$$

Apply condition (iv) in (4) we get

$$u\,(x\,,0)=\sum_{n=1}^{\infty}\mathrm{B_n}\sin\frac{n\pi}{a}x\left[\sin h\frac{n\pi b}{a}\right]=100$$

$$\sum_{n=1}^{\infty}\mathrm{B_n}\sin\frac{n\pi}{a}x\left[\sin h\frac{n\pi b}{a}\right]=100$$

To find B_n expand in half range Fourier sine series in the interval (0,a)

$$B_n\sin h\frac{n\pi b}{a}=\frac{2}{a}\int_0^a 100\sin\frac{n\pi}{a}x\ dx$$

$$= \frac{2}{a}\int_0^a 100 \sin\frac{n\pi}{a}x \ dx$$

$$= \frac{200}{a}\left[-\frac{\cos\frac{n\pi}{a}x}{\frac{n\pi}{a}}\right]_0^a$$

$$= -\frac{200}{n\pi}[\cos n\pi - \cos 0] = \frac{200}{n\pi}[1-(-1)^n]$$

$$B_n \sin h\frac{n\pi b}{a} = \begin{cases}\frac{400}{n\pi}, if\ n\ is\ odd\\ 0, if\ n\ is\ even\end{cases}$$

$$B_n = \begin{cases}\frac{400}{n\pi \sin h\frac{n\pi b}{a}}, if\ n\ is\ odd\\ 0, if\ n\ is\ even\end{cases}$$

Substitute B_n in (4)

$$u(x,y) = \sum_{n=odd}^{\infty}\frac{400}{n\pi\sin h\frac{n\pi b}{a}}\sin\frac{n\pi}{a}x\left[\sin h\frac{n\pi y}{a}\right]$$

22. A square plate is bounded by the lines $x=0, y=0, x=20 \ \& \ y=20$ Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x,0)=x(20-x), 0<x<20$ while the other two edges are kept at 0˚C. Determine the steady state temperature distribution in the plate.

Solution:

Let $u(x,y)$ be the temperature at any point (x,y).

Then $u(x,y)$ satisfies the Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x^2}=0$

The boundary Conditions are

$(i)\ u(0,20) = 0\ for\ 0 \le x \le 20$

$(ii)\ u(0,y) = 0\ for\ 0 \le y \le 20$

$(iii) u(20,y) = 0\ ,0<y<20$

$(iv)\ u(x,20) = x(20-x)\ ,0<y<20$

Now the suitable solution which satisfies our boundary conditions is given by

$$u(x,y) = (A\cos\lambda x + B\sin\lambda x)(c\,e^{\lambda y} + De^{-\lambda y}) \ldots\ldots\ldots\ldots(1)$$

Apply condition (i) in (1) we get

$$u(x,0) = (A\cos\lambda x + B\sin\lambda x)(C+D) = 0$$

$$(A\cos\lambda x + B\sin\lambda x) \ne 0$$

$$\therefore C+D=0$$

$$\Rightarrow D=-C$$

Substitute$D=-C\ in$ (1) we get

$$u(x,y) = (A\cos\lambda x + B\sin\lambda x)(c\,e^{\lambda y} - Ce^{-\lambda}\ ) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Apply condition $(ii)\ in$ (2) we get

$$u(0,y) = AC(\,e^{\lambda y} - e^{-\lambda}\ ) = 0$$

$$\Rightarrow A = 0$$

Substitute $A = 0$ in (2) we get

$$u(x,y) = (B\sin\lambda x)C\left(e^{\lambda y} - e^{-\lambda y}\right) \dots\dots\dots\dots\dots\dots\dots\dots (3)$$

Apply condition (iii) in (3) we get

$$u\,(20, Y) = (B\sin\lambda 20)C\left(e^{\lambda y} - e^{-\lambda y}\right) = 0$$

$$\sin\lambda 20 = 0$$

$$\sin\lambda 20 = \sin n\pi$$

$$\Rightarrow \lambda = \frac{n\pi}{20}$$

Substitute $\lambda = \frac{n\pi}{20}$ in (3) we get

$$u(x,y) = BC\ \sin\frac{n\pi}{20}x\left(e^{\frac{n\pi}{20}y} - e^{-\frac{n\pi}{20}y}\right)$$

$$= B\,\mathrm{D}\sin\frac{n\pi}{20}x\ 2\sin h\ \frac{n\pi y}{20}$$

$$u(x,y) = B_n\ \sin\frac{n\pi}{20}x\sin h\ \frac{n\pi y}{20}$$

The most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n\ \sin\frac{n\pi}{20}x\ \ \sin h\ \frac{n\pi y}{20} \dots\dots\dots\dots\dots\dots\dots (4)$$

Apply condition (iv) in (4) we get

$$u(x,20) = \sum_{n=1}^{\infty} B_n\ \sin\frac{n\pi}{20}x\ \ \sin h\ \frac{n\pi 20}{20} = x(20-x)$$

To find B_n expand in half range Fourier sine series in the interval (0,20)

$$B_n = \frac{2}{20}\int_0^{20} x(20-x)\sin\frac{n\pi}{20}x\ dx$$

$$= \frac{1}{10}\left[-\big(x(20-x)\big)\frac{\cos 20x}{\frac{n\pi}{20}} + (20-2X)\frac{\sin\frac{n\pi}{20}x}{\frac{n^2\pi^2}{20^2}} - 2\frac{\cos\frac{n\pi}{20}x}{\frac{n^3\pi^3}{20^3}}\right]_0^{20}$$

$$= \frac{1600}{n^3\pi^3}[1-(-1)^n]$$

$$B_n = \begin{cases} 0, if\ n\ iseven \\ \frac{3200}{n^3\pi^3}, if\ n\ is\ odd \end{cases}$$

Now substitute the value of B_n in (4) we get

$$u(x,y) = \sum_{n=odd}^{\infty} \frac{3200}{n^3\pi^3}\ \sin\frac{n\pi}{20}x\ \sin h\ n\pi$$

23. A rod 30 cm long has its ends A and B kept at 20^0 and 80^0 respectively until steady state conditions prevail the temperature at each end is then suddenly reduced to 0^0 and kept so. Find the resulting temperature function $u\,(x,t)$ taking $x = 0$ at A.

Solution:

The temperature function $u(x,t)$ is the solution of the one dimensional heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

the steady state conditions prevails $\frac{\partial u}{\partial t} = 0$

And hence we get $\frac{\partial^2 u}{\partial x^2} = 0$

On integration $u(x) = ax + b$

When $x = 0$ we get $u(0) = b$

When $x = 30$ we get $u(30) = 30a + 20b$

$$80 = 30a + 20b$$

$$60 = 30a$$

$$a = 2$$

Thus $u(x,0) = f(x) = 2x + 20$

The boundary conditions are

(i) $u(0,t) = 0 \text{ for all } t \geq 0$

(ii) $u(30,t) = 0 \text{ for all } t \geq 0$

(iii) $u(x,0) = f(x) = 2x + 20$

Now the suitable solution which satisfies our boundary conditions is

$$\boldsymbol{u(x,t) = (A\cos\lambda x + B\sin\lambda x)e^{-\lambda^2 c^2 t} \dots\dots\dots\dots (1)}$$

Apply condition (i) in (1) we get

$$u(0,t) = Ae^{-\lambda^2 c^2 t}$$

Here $e^{-\lambda^2 c^2 t} \neq 0$ ($\because$ *It is defined for all t*) $\therefore$ **A = 0**

Substitute $A = 0$ in (1) we get

$$u(x,t) = B\sin\lambda x\ e^{-\lambda^2 c^2 t} \dots\dots\dots\dots (2)$$

Apply condition (ii) in (2) we get

$$u(30,t) = B\sin 30\lambda\ e^{-\lambda^2 c^2 t} = 0$$

Here $e^{-\lambda^2 c^2 t} \neq 0$ ($\because$ *It is defined for all t*)

Also $B \neq 0$, ($\because$ *we have* $A = 0$, *if* $B = a$ *then we get trivial solution*)

$$\therefore \sin 30\lambda = 0$$

$$\sin 30\lambda = \sin n\pi = 0$$

$$30\lambda = n\pi$$

$$\boldsymbol{\lambda = \frac{n\pi}{30}}$$

Substitute $\lambda = \frac{n\pi}{30}$ in (2) we get

$$u(x,t) = B\sin\frac{n\pi}{30}x\ e^{-\frac{n^2\pi^2}{900}c^2 t} \dots\dots\dots\dots (3)$$

The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{30}x\ e^{-\frac{n^2\pi^2}{900}c^2 t} \dots\dots\dots\dots (4)$$

Apply condition (iii) in (4) we get

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{30}x\ e^0 = 2x + 20$$

To find B_n expand $2x+20$ in a half range Fourier sine series in the interval (0 , 30)

$$B_n = \frac{2}{30}\int_0^{30} (2x+20)\ \sin\frac{n\pi}{30}x\,dx$$

$$= \frac{1}{15}\int_0^{30} (2x+20)\ \sin\frac{n\pi}{30}x\,dx$$

Using Bernoulli formula $\int uvdx = uv_1 - u'v_2 + u''v_3 - \ldots$

$u = (20x - x^2)$	$v = \sin\frac{n\pi x}{20}$
$u' = (20 - 2x)$	$v_1 = -\frac{\cos\frac{n\pi x}{20}}{\frac{n\pi}{20}}$
$u'' = (-2)$	$v_2 = -\frac{\sin\frac{n\pi x}{20}}{\frac{n^2\pi^2}{20^2}}$
$u''' = 0$	$v_3 = +\frac{\cos\frac{n\pi x}{20}}{\frac{n^3\pi^3}{20^3}}$

$$= \frac{1}{15}\left[-(2x+20)\frac{\cos\frac{n\pi}{30}x}{\frac{n\pi}{30}} + 2\frac{\sin\frac{n\pi}{30}x}{\frac{n^2\pi^2}{900}}\right]_0^{30}$$

$$= \frac{1}{15}\left[\frac{-2400}{n\pi}\cos n\pi + 0 + \frac{600}{n\pi} - 0\right]$$

$$= \frac{600}{15n\pi}[-4(-1)^n + 1]$$

$$B_n = \frac{40}{n\pi}[1 - 4(-1)^n]$$

Substitute the value of B_n in (4) we get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}[1-4(-1)^n]\sin\frac{n\pi}{30}x\ e^{-\frac{n^2\pi^2}{900}c^2t}$$

$$u(x,t) = \frac{40}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}[1-4(-1)^n]\sin\frac{n\pi}{30}x\ e^{-\frac{n^2\pi^2}{900}c^2t}$$

UNIT-IV

PART A

1. State Fourier integral theorem.

Solution:

If f(x) is piece-wise continuously differentiable and absolutely integrable in$(-\infty,\infty)$, then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{is(x-t)}\, dt\, ds$$

(OR)

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\, dt\, d\lambda$$

This is known as Fourier integral theorem or Fourier integral formula.

2. State Parseval's identity on Fourier Transform.

Solution:

If F(s) is the Fourier transform of f(x) then

$$\int_{-\infty}^{\infty}|f(x)|^2 dx = \int_{-\infty}^{\infty}|F(s)|^2 ds$$

3. Prove that the FCT of $\{f(x)\cos ax\}$ is given by $\frac{1}{2}[F_c(s+a)+F_c(s-a)]$ where $F_c(s)$ is the Fourier cosine transform of $f(x)$.

Solution:

We know that

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\, dx$$

$$F_c[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos ax\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\cos ax\, dx$$

$$F_c[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\frac{1}{2}[\cos(s+a)x + \cos(s-a)x]\, dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(s+a)x\, dx + \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(s-a)x]dx\right]$$

$$F_c[f(x)\cos ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$$

4. If $F(s)$ is the Fourier transform of $f(x)$, show that $F\{f(x-a)\} = e^{ias}F(s)$.

Solution:

We know that $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)e^{isx}dx$$

Let $x - a = t$

$dx = dt$

$x \to -\infty \quad \Rightarrow t \to -\infty$

$x \to \infty \quad \Rightarrow t \to \infty$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{is(t+a)} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} e^{isa} dt$$

$$= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt$$

$$F[f(x-a)] = e^{isa} F(s)$$

5. Find the Fourier cosine transform of $e^{-ax}, a > 0$.

Solution:

Given $f(x) = e^{-ax}$

We know that

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx$$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx \, dx$$

Using the formula $\int_0^{\infty} e^{-a} \text{ cossx } dx = \frac{a}{a^2+s^2}$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+s^2}\right]$$

6. If $F_c(s) = f(x)$, **prove that** $F_c\{f(ax)\}$ **is** $\frac{1}{a} F_c\left(\frac{s}{a}\right)$.

Solution:

We know that Fourier cosine transform formula is

$$F_c[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) cossx dx$$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) cossx dx$$

$$F[f(ax)] = R.P \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{isx} dx$$

Let $ax = t \quad \Rightarrow a\, dx = dt$

$x \to -\infty \quad \Rightarrow t \to -\infty, \quad x \to \infty \quad \Rightarrow \quad t \to \infty$

$$F[f(ax)] = R.P \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{\frac{ist}{a}} \frac{dt}{a}$$

$$F[f(ax)] = R.P \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{is\left(\frac{t}{a}\right)} dt$$

$$F[f(ax)] = \frac{1}{a} F_c\left(\frac{s}{a}\right)$$

7. State Fourier Transform pair.

Solution:

The Fourier Transform of f(x) is given by

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

The inverse Fourier transform of F(s) is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

8.

tate and prove the change of scale property on Fourier Transform.

Solution:

We know that

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{isx} dx$$

Let $ax = t \quad \Rightarrow \quad a\,dx$

$x \to -\infty \quad \Rightarrow t \to -\infty$

$x \to \infty \quad \Rightarrow \ t \to \infty$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{\frac{ist}{a}} \frac{dt}{a}$$

$$F[f(ax)] = \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{is\left(\frac{t}{a}\right)} dt$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\left(\frac{s}{a}\right)x} dx$$

$$F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right) \qquad \ldots\ldots\ldots\ldots\ldots \ (1)$$

Similarly if $a < 0$

S

$$F[f(ax)] = \frac{1}{-a} F\left(\frac{s}{a}\right) \qquad \ldots\ldots\ldots\ldots \ (2)$$

From (1) & (2)

$$F[f(ax)] = \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

9. State Convolution Theorem on Fourier Transform.

Solution:

The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

$$F[f(x) * g(x)] = F(s)G(s) = F[f(x)]F[g(x)]$$

10. State and prove modulation Theorem on Fourier Transform.

Solution:

Statement:

If F(s) is the Fourier transform of f(x), then $F[f(x)cos\ ax] = \frac{1}{2}[F(s+a)+F(s-a)]$

Proof:

We know that $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos ax\, e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left[\frac{e^{iax}+e^{-iax}}{2}\right]e^{isx}dx$$

$$F[f(x)\cos ax] = \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left[e^{i(s+a)x}+e^{i(s-a)x}\right]dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s+a)x}dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s-a)x}dx\right.$$

$$F[f(x)\cos ax] = \frac{1}{2}[F(s+a)+F(s-a)]$$

11. **Define self reciprocal with respect to Fourier Transforms.**
Solution:
If $F\{f(x)\}$ is f(s) , then $f(x)$ is self reciprocal under Fourier transform.

12. **Define Fourier sine transform and its inversion formula.**
Solution:

The infinite Fourier sine transforms of $f(x)$ is defined by

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\, dx$$

The inverse Fourier sine transform is defined by

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s[f(x)]\sin sx\, ds$$

13. **Find the Fourier transform $f(x)$ defined by** $f(x)=\begin{cases}0\ ,\ x<a\\ 1\ ,\ a<x<b\\ 0\ ,\ x>b\end{cases}$

Solution:

We know that $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$= \frac{1}{\sqrt{2\pi}}\int_a^b 1.e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{isx}}{is}\right)_a^b$$

$$= \frac{1}{is\sqrt{2\pi}}\left(e^{isb}-e^{isa}\right)$$

14. **Find the Fourier sine transform of** $f(x)=\frac{1}{x}$.

Solution:

We know that

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\sin sx\,dx$$

Let $sx = \theta$

$s\,dx = d\theta$

$x \to 0 \quad \Rightarrow \theta \to 0$

$x \to \infty \quad \Rightarrow \theta \to \infty$

$$F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{s}{\theta}\sin\theta\,\frac{d\theta}{s}$$

$$F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin\theta}{\theta}\,d\theta$$

$$F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{\pi}{2}\right] = \sqrt{\frac{\pi}{2}} \qquad since \int_0^\infty \frac{\sin\theta}{\theta}\,d\theta = \frac{\pi}{2}$$

15. Find the Fourier transform of $e^{-\alpha|x|}, \alpha > 0$.

Solution:

Given $f(x) = e^{-a|x|}$

$$f(x) = \begin{cases} e^{\alpha x}, if -\infty < x < 0 \\ e^{-\alpha x}, if\ 0 < x < \infty \end{cases}$$

W.K.T $\quad F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha|x|}e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha|x|}(cossx + isinsx)dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-\alpha x}cossxdx$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\left[\frac{\alpha}{\alpha^2+s^2}\right]$$

$\because \int_0^\infty e^{-ax}\cos sx\ dx = \frac{a}{a^2+s^2} \qquad$ here $a = \alpha$

$$F[f(x)] = \sqrt{\frac{2}{\pi}}\left[\frac{\alpha}{\alpha^2+s^2}\right]$$

PART B

16. Determine the Fourier transform of $f(x)$ **if** $f(x) = \begin{cases} 1, & if\ |x| < 1 \\ 0, & if\ |x| > 1 \end{cases}$.

Hence deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$.

Solution:

We know that

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1\,(\cos sx + \sin sx)\, dx$$

$\sin sx$ is an odd function $\frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \sin sx\, dx = 0$

$$F[f(x)] = \frac{2}{\sqrt{2\pi}} \int_{0}^{1} \cos\ sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \left[\frac{\sin sx}{s}\right]_0^1$$

$$F(s) = \frac{2}{\sqrt{2\pi}} \left(\frac{\sin s}{s}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \ (1)$$

This is the required Fourier transforms.

i) By inverse Fourier transform formula

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx}\, ds$$

$$1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}} \left(\frac{\sin s}{s}\right) (cossx - isinsx)\, ds$$

$$1 = \frac{2}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s}{s}\right) (\cos sx - i \sin sx)\, ds$$

Equating real part on both sides

$$1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s}{s}\right) \cos sx\ \ ds$$

$$1 = \frac{2}{\pi} \int_{0}^{\infty} \left(\frac{\sin s}{s}\right) \cos sx\ \ ds + 0$$

$$\int_{0}^{\infty} \left(\frac{\sin s}{s}\right) \cos sx\ \ ds = \frac{\pi}{2}$$

Putting x = 0, we get

$$\int_0^\infty \left(\frac{\sin s}{s}\right) cos0\, ds = \frac{\pi}{2}$$

$$\int_{0}^{\infty} \left(\frac{\sin s}{s}\right) ds = \frac{\pi}{2} \qquad since \cos 0 = 1$$

$$\int_{0}^{\infty} \left(\frac{\sin t}{t}\right) dt = \frac{\pi}{2}$$

ii) By Parseval's identity

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

From (1)

$$\therefore \int_{-\infty}^{\infty} \left(\frac{2}{\sqrt{2\pi}}\left(\frac{\sin s}{s}\right)\right)^2 ds = \int_{-1}^{1} 1dx$$

$$\frac{4}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = [x]_{-1}^{1} = 2$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = 2$$

$$\int_{-\infty}^{\infty} \left(\frac{\sin s}{s}\right)^2 ds = 2 \times \frac{\pi}{2} = \pi$$

$$2\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi$$

$$\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

17. Construct the Fourier transform of $f(x) = \begin{cases} a^2 - x^2 , & if \ |x| < a \\ 0 , & if \ |x| > a > 0 \end{cases}$. **Hence deduce that** $\int_0^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)dt = \frac{\pi}{4}$ **.Using Parseval's identity, show that** $\int_0^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$.

Solution:

The given function can be written as

$$f(x) = \begin{cases} a^2 - x^2, if \ -a < x < a \\ 0, \quad otherwise \end{cases}$$

We know that

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\, dx$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} (a^2 - x^2)(\cos sx + \sin sx)\, dx$$

$$F[f(x)] = \frac{2}{\sqrt{2\pi}}\int_{0}^{a} (a^2 - x^2)(\cos sx)\, dx$$

Using Bernoulli formula $\int uv\, dx = uv_1 - u'v_2 + u''v_3 - \cdots$

Here

$u = a^2 - x^2 \qquad v = \cos sx$

$u' = -2x \qquad v_1 = \frac{\sin sx}{s}$

$$u'' = -2 \qquad v_2 = \frac{-\cos sx}{s^2}$$

$$v_3 = -\frac{\sin sx}{s^3}$$

$$F[f(x)] = \frac{2}{\sqrt{2\pi}}\left[(a^2 - x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(\frac{-\cos sx}{s^2}\right) + 2\frac{\sin sx}{s^3}\right]_0^a$$

$$F[f(x)] = \frac{2}{\sqrt{2\pi}}\left[-\frac{2a\cos as}{s^2} + \frac{2\sin as}{s^3}\right]$$

$$F[s] = \frac{4}{\sqrt{2\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \ (1)$$

i)By inverse Fourier transform formula

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$$

$$a^2 - x^2 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{4}{\sqrt{2\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)e^{-isx}\,ds$$

$$a^2 - x^2 = \frac{4}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)(\cos sx - i\sin sx)\,ds$$

Equating real part on both sides

$$a^2 - x^2 = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\cos sx\,ds$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\cos sx\,ds = a^2 - x^2$$

$$\frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\cos sx\,ds = a^2 - x^2$$

Let $a = 1$ & $x = 0$, we get

$$\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos(0)\,ds = \frac{\pi}{4}\quad(1-0)$$

$$\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)ds = \frac{\pi}{4}$$

Let $s = t \Rightarrow ds = dt$

$$\int_{0}^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)dt = \frac{\pi}{4}$$

ii) By Parseval's identity

$$\int_{-\infty}^{\infty}|F(s)|^2\,ds = \int_{-\infty}^{\infty}|f(x)|^2\,dx$$

From (1)

$$\therefore \int_{-\infty}^{\infty}\left(2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)\right)^2 ds = \int_{-a}^{a}(a^2 - x^2)^2 dx$$

$$\int_{-\infty}^{\infty}\left(\frac{4}{\sqrt{2\pi}}\left(\frac{\sin as - as\cos a}{s^3}\right)\right)^2 ds = \int_{-1}^{1}(1 - x^2)^2 dx$$

Put $a = 1$ we get

$$\frac{16}{2\pi} \times 2\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\int_{0}^{1}(1 - 2x^2 + x^4)dx$$

$$\frac{16}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_0^1$$

$$\frac{16}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\left[\left(1 - \frac{2}{3} + \frac{1}{5}\right) - (0 - 0 + 0)\right]$$

$$= 2\left[\frac{15 - 10 + 3}{15}\right] = \frac{16}{15}$$

$$\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{16}{15}\left(\frac{\pi}{16}\right) = \frac{\pi}{15}$$

Let $s = t \Rightarrow ds = dt$

$$\int_{0}^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$$

18. Find the Fourier transform of $f(x) = \begin{cases} 1 - |x|, & if \quad |x| < 1 \\ 0, & if \quad |x| > 1 \end{cases}$ **and hence find the value of** $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt$.

Solution:

The given function can be written as

$$f(x) = \begin{cases} 1 - |x|, if & -1 < x < 1 \\ 0, & otherwise \end{cases}$$

We know that

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\, dx$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1 - |x|)e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1 - |x|)(\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1 - |x|)\cos sx\, dx + \frac{i}{\sqrt{2\pi}}\int_{-1}^{1}(1 - |x|)\sin sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1 - |x|)\cos sx\, dx + 0$$

Since $(1-|x|)\sin sx$ is odd function

$$= \frac{2}{\sqrt{2\pi}}\int_0^1 (1-x)\cos sx\,dx$$

Using Bernoulli formula $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$

Here

$u = 1-x \quad v = \cos sx$

$u' = -1 \quad v_1 = \frac{\sin sx}{s}$

$v_2 = \frac{-\cos sx}{s^2}$

$$= \frac{2}{\sqrt{2\pi}}\left[(1-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^1$$

$$= \frac{2}{\sqrt{2\pi}}\left[(1-x)\left(\frac{\sin sx}{s}\right) - \frac{\cos sx}{s^2}\right]_0^1$$

$$= \frac{2}{\sqrt{2\pi}}\left[\left(0-\frac{\cos s}{s^2}\right) - \left(0-\frac{1}{s^2}\right)\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[-\frac{\cos s}{s^2} + \frac{1}{s^2}\right]$$

$$F[s] = \frac{2}{\sqrt{2\pi}}\left[\frac{1-\cos s}{s^2}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (1)$$

$$F[s] = \frac{2}{\sqrt{2\pi}}\left[\frac{2sin^2\left(\frac{s}{2}\right)}{s^2}\right] \qquad since \quad 1-\cos s = \quad 2sin^2\left(\frac{s}{2}\right)$$

$$F[s] = \frac{4}{\sqrt{2\pi}}\left[\frac{sin^2\left(\frac{s}{2}\right)}{s^2}\right]$$

This is the required Fourier Transforms

i) By inverse Fourier transform formula ,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$$

$$1-|x| = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{4}{\sqrt{2\pi}}\left[\frac{sin^2\left(\frac{s}{2}\right)}{s^2}\right]e^{-isx}\,ds$$

$$1-x = \frac{4}{2\pi}\int_{-\infty}^{\infty} \frac{sin^2\left(\frac{s}{2}\right)}{s^2}(\cos sx - i\sin sx)\,ds$$

Equating real part on both sides

$$1-x = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{sin^2\left(\frac{s}{2}\right)}{s^2}\right)\cos sx \quad ds$$

$$1-x=\frac{2}{\pi}\times 2\int_0^\infty \left(\frac{sin^2\left(\frac{s}{2}\right)}{s^2}\right)\cos sx\ ds$$

$Let\ x=0$

$$1=\frac{4}{\pi}\int_0^\infty \left(\frac{sin^2\left(\frac{s}{2}\right)}{s^2}\right)\cos s(0)\ ds \Rightarrow 1=\frac{4}{\pi}\int_0^\infty \left(\frac{sin^2\left(\frac{s}{2}\right)}{s^2}\right) ds$$

$Let\ \frac{s}{2}=t$

$s=2t$

$ds=2dt$

$$1=\frac{4}{\pi}\int_0^\infty \left(\frac{sin^2 t}{4t^2}\right) 2\,dt$$

$$1=\frac{8}{\pi}\int_0^\infty \left(\frac{sin^2 t}{4t^2}\right) dt$$

$$\frac{2}{\pi}\int_0^\infty \left(\frac{sin^2 t}{t^2}\right) dt=1$$

$$\int_0^\infty \left(\frac{sin^2 t}{t^2}\right) dt=\frac{\pi}{2}$$

ii) By Parseval's identity

$$\int_{-\infty}^{\infty}|F(s)|^2\,ds = \int_{-\infty}^{\infty}|f(x)|^2\,dx$$

From (1) $\therefore \int_{-\infty}^{\infty}\left(\frac{4}{\sqrt{2\pi}}\left[\frac{sin^2\left(\frac{s}{2}\right)}{s^2}\right]\right)^2 ds=\int_{-1}^{1}(1-|x|)^2dx$

$$\frac{16}{2\pi}\times 2\int_0^\infty \left(\frac{sin^2\left(\frac{s}{2}\right)}{s^2}\right)^2 ds = 2\int_0^1 (1-x)^2dx$$

$$=2\left[\frac{-(1-x)^3}{3}\right]_0^1=-\frac{2}{3}[(1-x)^3]_0^1$$

$$\frac{16}{\pi}\int_{-\infty}^{\infty}\frac{sin^4\left(\frac{s}{2}\right)}{s^4}ds=-\frac{2}{3}[0-1]=\frac{2}{3}$$

$$\int_0^\infty \frac{sin^4\left(\frac{s}{2}\right)}{s^4}ds=\frac{2}{3}\times\frac{\pi}{16}$$

$$\int_0^\infty \frac{sin^4\left(\frac{s}{2}\right)}{s^4}ds=\frac{\pi}{24}$$

$Let\quad \frac{s}{2}=t$

$$s = 2t$$

$$ds = 2dt$$

$$\int_0^\infty \left(\frac{sin^2 t}{(2t)^2}\right)^2 2\,dt = \frac{\pi}{24}$$

$$\frac{1}{8}\int_0^\infty \left(\frac{sin^2 t}{t^2}\right)^2 dt = \frac{\pi}{24}$$

$$\int_0^\infty \left(\frac{sin^2 t}{t^2}\right)^2 dt = \frac{\pi}{24} \times 8$$

$$\int_0^\infty \frac{sin^4 t}{t^4} dt = \frac{\pi}{3}$$

19. Show that $e^{-x^2/2}$ is self reciprocal with respect to Fourier transform.

Solution:

We know that

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - is\ )} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[(ax)^2 - isx]} dx \qquad \dots (1)$$

We know that

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$a^2 - 2ab = (a-b)^2 - b^2$$

$$Here\ a = ax, \qquad 2ab = isx$$

$$2(ax)b = isx$$

$$2ab = is$$

$$b = is/2a$$

$$(ax)^2 - isx = \left(ax - \frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2$$

$$= \left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}$$

$$e^{-[(ax)^2 - isx]} = e^{-\left[\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right]}$$

$$= e^{-\left[\left(ax - \frac{is}{2a}\right)^2\right]} e^{-\left[\frac{s^2}{4a^2}\right]}$$

Substituting in equation (1) we get

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2\right]} e^{-\left[\frac{s^2}{4a^2}\right]} dx$$

$$F(s) = \frac{e^{-\left[\frac{s^2}{4a^2}\right]}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2\right]} dx \qquad \ldots(2)$$

Put $ax - \frac{is}{2a} = u$

$a\,dx = du$

$x \to -\infty \quad \Rightarrow u \to -\infty$

$x \to \infty \quad \Rightarrow u \to \infty$

$$F(s) = \frac{e^{-\left[\frac{s^2}{4a^2}\right]}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \frac{1}{a}\, du$$

$$F(s) = \frac{e^{-\left[\frac{s^2}{4a^2}\right]}}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2}\, du \qquad \ldots(3)$$

$$= \frac{e^{-\left[\frac{s^2}{4a^2}\right]}}{a\sqrt{2\pi}} \sqrt{\pi}$$

$$F(s) = \frac{e^{-\left[\frac{s^2}{4a^2}\right]}}{a\sqrt{2}}$$

$$F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}}$$

Put $a = \frac{1}{\sqrt{2}}$ we get

$$F\left[e^{-x^2/2}\right] = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)\sqrt{2}} e^{-\frac{s^2}{4\left(\frac{1}{2}\right)}} = e^{-s^2/2}$$

$$F\left[e^{-x^2/2}\right] = e^{-s^2/2}$$

20. State and prove Convolution Theorem on Fourier Transform.

Solution:

Statement:

The Fourier transform of the convolution of f(x) and g(x) is the product of their Fourier transforms.

$$\boldsymbol{F[f(x) * g(x)] = F(s)G(s) = F[f(x)]F[g(x)]}$$

Proof:

We know that

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[f(x) * g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f(x) * g(x)] e^{isx} dx$$

$$F[f(x) * g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t) dt\right] e^{isx} dx$$

$$F[f(x)*g(x)] = \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)g(x-t)e^{isx}dt\,dx$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)g(x-t)e^{isx}dx\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-t)\,e^{isx}dx\right]dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,F[g(x-t)]dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{ist}G(s)dt$$

$$[\because F[f(x-a)] = e^{-ias}F(s)]$$

$$= G(s)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}dt\right]$$

$$F[f(x)*g(x)] = G(s)F(s) = F(s)G(s)$$

21. State and prove Parseval's identity on transforms.

Solution:

Statement:

If F(s) is the Fourier transform of f(x) then $\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|F(s)|^2ds$

Proof:

By the Convolution theorem

$$F[f(x)*g(x)] = F(s).G(s)$$

$$f(x)*g(x) = F^{-1}[F(s).G(s)]$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,g(x-t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\,G(s)\,e^{-is}\;ds$$

Put x = 0 and $g(-t) = \overline{f(t)}$ there it follows that $G(s) = \overline{F(s)}$

$$\int_{-\infty}^{\infty} f(t)\overline{f(t)}\,dt = \int_{-\infty}^{\infty} F(s)\overline{F(s)}\,ds$$

$$\int_{-\infty}^{\infty}|f(t)|^2dt = \int_{-\infty}^{\infty}|F(s)|^2ds$$

Put $t = x \;\Rightarrow dt = dx$

$$\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|F(s)|^2ds$$

22. Find the Fourier cosine transform of $e^{-a^2x^2}$ for any $a>0$ and hence prove that $e^{-x^2/2}$ is self reciprocal under Fourier cosine transform.

Solution:

We know that

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\,dx$$

$$F_c\left[e^{-a^2x^2}\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-a^2x^2}\cos sx\,dx$$

$$F_c\left[e^{-a^2x^2}\right] = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2}\cos sx\,dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\,R.P\int_{-\infty}^{\infty} e^{-a^2x^2}e^{isx}dx$$

$$F_c\left[e^{-a^2x^2}\right] = \frac{1}{\sqrt{2\pi}}R.P\int_{-\infty}^{\infty} e^{-a^2x^2}e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}R.P\int_{-\infty}^{\infty} e^{-a^2x^2+isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}R.P\int_{-\infty}^{\infty} e^{-(a^2x^2-isx)}dx \qquad \ldots(1)$$

We know that

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$a^2 - 2ab = (a-b)^2 - b^2$$

$$Here\ a = ax, \qquad 2ab = isx$$

$$2(ax)b = isx$$

$$2ab = is$$

$$b = is/2a$$

$$(ax)^2 - isx = \left(ax - \frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2$$

$$= \left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}$$

$$e^{-[(ax)^2-isx]} = e^{-\left[\left(ax\ \frac{is}{2a}\right)^2+\frac{s^2}{4a^2}\right]}$$

$$= e^{-\left[\left(ax-\frac{is}{2a}\right)^2\right]}e^{-\left[\frac{s^2}{4a^2}\right]}$$

Substituting in equation (1) we get

$$F_c\left[\boldsymbol{e^{-a^2x^2}}\right] = \frac{1}{\sqrt{2\pi}}R.P\int_{-\infty}^{\infty} e^{-\left[\left(ax-\frac{is}{2a}\right)^2\right]}e^{-\left[\frac{s^2}{4a^2}\right]}dx$$

$$F_c\left[\boldsymbol{e^{-a^2x^2}}\right] = \frac{e^{-\left[\frac{s^2}{4a^2}\right]}}{\sqrt{2\pi}}R.P\int_{-\infty}^{\infty} e^{-\left[\left(ax-\frac{is}{2a}\right)^2\right]}dx \qquad \ldots(2)$$

Put $ax - \frac{is}{2a} = u$

$$a\,dx = du$$

$$x \to -\infty \quad \Rightarrow u \to -\infty \qquad x \to \infty \quad \Rightarrow u \to \infty$$

$$F_c\left[e^{-a^2x^2}\right] = \frac{e^{-\left[\frac{s^2}{4a^2}\right]}}{\sqrt{2\pi}} R.P \int_{-\infty}^{\infty} e^{-u^2} \frac{1}{a}\, du$$

$$F_c\left[e^{-a^2x^2}\right] = \frac{e^{-\left[\frac{s^2}{4a^2}\right]}}{a\sqrt{2\pi}} R.P \int_{-\infty}^{\infty} e^{-u^2}\, du \qquad \ldots(3)$$

$$= \frac{e^{-\left[\frac{s^2}{4a^2}\right]}}{a\sqrt{2\pi}} \sqrt{\pi}$$

$$F_c\left[e^{-a^2x^2}\right] = \frac{e^{-\left[\frac{s^2}{4a^2}\right]}}{a\sqrt{2}}$$

$$F_c\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}}$$

Put $a = \frac{1}{\sqrt{2}}$ we get

$$F_c\left[e^{-x^2/2}\right] = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)\sqrt{2}} e^{-\frac{s^2}{4\left(\frac{1}{2}\right)}} = e^{-s^2/2}$$

$$F_c\left[e^{-x^2/2}\right] = e^{-s^2/2}$$

Hence $f(x) = e^{-x^2/2}$ is self reciprocal with respect to Fourier Cosine transform.

23. Find the Fourier cosine and sine transform of $f(x) = e^{-ax}, a > 0$ and hence deduce the inversion formula.

Solution:

We know that

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx\, dx$$

$$F_c[e^{-a}\ ] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx\, dx$$

$$F_c[e^{-a}\ ] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+s^2}\right]$$

Applying the inverse formula, we have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c[e^{-ax}] \cos sx\ ds$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+s^2}\right] \cos sx\ ds$$

$$f(x) = \frac{2a}{\pi}\int_0^\infty \frac{\cos sx}{a^2+s^2}ds$$

$$\int_0^\infty \frac{\cos sx}{a^2+s^2}ds = \frac{\pi}{2a}f(x)$$

$$\int_0^\infty \frac{\cos sx}{a^2+s^2}ds = \frac{\pi}{2a}\ e^{-ax}\ , a>0$$

Put $s = x$ and $x = m$

$$\int_0^\infty \frac{\cos mx}{a^2+x^2}dx = \frac{\pi}{2a}\ e^{-am}\ , a>0$$

We know that

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\ dx$$

$$F_s[e^{-a}\ ] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\ dx$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]$$

Applying the inverse formula, we have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s[e^{-ax}]\sin sx\ ds$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]\sin sx\ ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{s\ \sin sx}{a^2+s^2}ds$$

$$\int_0^\infty \frac{s\sin sx}{a^2+s^2}ds = \frac{\pi}{2}f(x)$$

$$\int_0^\infty \frac{s\sin sx}{a^2+s^2}ds = \frac{\pi}{2}\ e^{-ax}\ , a>0$$

Put $s = x \Rightarrow ds = dx$ *and* $x = m$

$$\int_0^\infty \frac{x\sin mx}{a^2+x^2}dx = \frac{\pi}{2}\ e^{-am}\ , a>0$$

24. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ **using Fourier cosine transform of** e^{-ax} **and** e^{-bx}**.**

Solution:

Let $f(x) = e^{-ax}$

W.K.T $F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\, dx$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\, dx$$

$$F_c[f(x)] = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right]$$

Let $g(x) = e^{-bx}$

We know that

$$F_c[g(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos sx\, dx$$

$$F_c[e^{-bx}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-bx}\cos sx\, dx$$

$$F_c[g(x)] = F_c[e^{-bx}] = \sqrt{\frac{2}{\pi}}\left[\frac{b}{s^2+b^2}\right]$$

By formula

$$\int_0^\infty F_c[f(x)]\, F_c[g(x)]\, ds = \int_0^\infty f(x)g(x)dx =$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right]\sqrt{\frac{2}{\pi}}\left[\frac{b}{s^2+b^2}\right] ds = \int_0^\infty e^{-ax}e^{-bx}dx$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)}ds = \int_0^\infty e^{-(a+b)x}dx$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)}ds = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)}ds = \left[0-\left(\frac{1}{-(a+b)}\right)\right]$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)}ds = \frac{1}{a+b}$$

$$\int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)}ds = \frac{\pi}{2ab(a+b)}$$

put $s = x \Rightarrow ds = dx$

$$\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{2ab(a+b)}$$

25. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$ using Parseval's identity.

Solution:

Let $f(x) = e^{-ax}$

We know that

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c[e^{-a}\ ] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$F_c[f(x)] = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right]$$

By Parseval's identity

$$\int_0^\infty |F_c[f(x)]|^2\,ds = \int_0^\infty |f(x)|^2\,dx$$

$$\int_0^\infty \left[\sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right]\right]^2 ds = \int_0^\infty (e^{-ax})^2\,dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{a^2}{(s^2+a^2)^2}\,ds = \int_0^\infty e^{-2ax}\,dx$$

$$\frac{2a^2}{\pi}\int_0^\infty \frac{1}{(s^2+a^2)^2}\,ds = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$= 0-\left(\frac{1}{-2a}\right) \quad = \frac{1}{2a}$$

$Put\ s = x \ \Rightarrow ds = dx$

$$\int_0^\infty \frac{1}{(x^2+a^2)^2}\,ds = \frac{\pi}{4a^3}$$

26. Find the Fourier Integral representation of $f(x)$ defined as

$$f(x)=\begin{cases} 0 & for\ \ x<0 \\ \frac{1}{2} & for\ \ x=0 \\ e^{-x} & for\ x>0 \end{cases}$$

Solution:

We know that the Fourier integral of $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)cos\lambda(t-x)dxd\lambda \ldots\ldots.(1)$$

$$\text{Here } f(t) = \begin{cases} 0 & for\ \ t<0\ \ ie. -\infty<t<0 \\ \frac{1}{2} & for\ \ \ t=0\ \ ie. \qquad t=0 \\ e^{-t} & for\ \ t>0\ \ ie.\ \ 0<t<\infty \end{cases}$$

$$f(x) = \frac{1}{\pi}\int_0^\infty\int_0^\infty e^{-t}cos\lambda(t-x)dtd\lambda$$

$$f(x) = \frac{1}{\pi}\int_0^\infty\int_0^\infty e^{-t}cos(\lambda t-\lambda x)dtd\lambda$$

$$= \frac{1}{\pi}\int_0^\infty\int_0^\infty e^{-t}(cos\lambda t\ cos\lambda x + sin\lambda t\ sin\lambda x)dtd\lambda$$

$$= \frac{1}{\pi}\int_0^\infty\int_0^\infty [e^{-t}\cos\lambda t \cos\lambda x + e^{-t}\sin\lambda t \sin\lambda x]dt d\lambda$$

$$= \frac{1}{\pi}\left[\int_0^\infty \cos\lambda x \int_0^\infty e^{-t}\cos\lambda t\, dt + \int_0^\infty \sin\lambda x \int_0^\infty e^{-t}\sin\lambda t\, dt\right]d\lambda$$

$$\because \int_0^\infty e^{-at}\cos bt = \frac{a}{a^2+b^2} \quad \& \quad \int_0^\infty e^{-at}\sin bt = \frac{b}{a^2+b^2}$$

$$= \frac{1}{\pi}\left[\int_0^\infty \cos\lambda x\left[\frac{1}{1+\lambda^2}\right] + \sin\lambda x\left[\frac{\lambda}{1+\lambda^2}\right]\right]d\lambda$$

$$= \frac{1}{\pi}\left[\int_0^\infty \left[\frac{\cos\lambda x}{1+\lambda^2}\right] + \left[\frac{\lambda\sin\lambda x}{1+\lambda^2}\right]\right]d\lambda$$

$$= \frac{1}{\pi}\left[\int_0^\infty \frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\right]d\lambda$$

$$\Rightarrow f(x) = \frac{1}{\pi}\left[\int_0^\infty \frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}\right]d\lambda$$

$$\Rightarrow \int_0^\infty \frac{\cos\lambda x + \lambda\sin\lambda x}{1+\lambda^2}d\lambda = \pi f(x)$$

27. Find the Fourier sine transform of $f(x) = \begin{cases} x & in\ 0<x<1 \\ 2-x & in\ 1<x<2 \\ 0 & in\ x>2 \end{cases}$

Solution:

We know that the Fourier sine transform of $f(x)$ is

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx dx$$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\left[\int_0^1 f(x)\sin sx dx + \int_1^2 f(x)\sin sx dx + \int_2^\infty f(x)\sin sx dx\right]$$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\left[\int_0^1 x\sin sx dx + \int_1^2 (2-x)\sin sx dx + 0\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left[x\left(\frac{-\cos sx}{s}\right) - 1\left(\frac{-\sin sx}{s^2}\right)\right]_0^1 + \left[(2-x)\left(\frac{-\cos sx}{s}\right) - (-1)\left(\frac{-\sin sx}{s^2}\right)\right]_1^2\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left[\frac{-x\cos sx}{s} + \frac{-\sin sx}{s^2}\right]_0^1 + \left[-(2-x)\left(\frac{\cos sx}{s}\right) - \left(\frac{\sin sx}{s^2}\right)\right]_1^2\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left[\frac{-\cos s}{s} + \frac{-\sin}{s^2} - (-0+0)\right] + \left[-0 - \frac{\sin 2s}{s^2} - \left(\frac{-co}{s}\right) - \left(\frac{\sin s}{s^2}\right)\right]\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left[\frac{-\cos s}{s} + \frac{\sin s}{s} - \frac{\sin 2s}{s^2} + \frac{\cos s}{s} + \frac{\sin s}{s^2}\right]\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{2\sin s - \sin 2}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{2\sin s - 2\sin s\cos}{s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} 2\, sin\, s \left[\frac{1-cos}{s^2}\right]$$

28. Solve for $f(x)$ from the equation $\int_0^\infty f(x)\sin sx\,dx = \begin{cases} 1, 0 \le s < 1 \\ 2, 1 \le s < 2 \\ 0, s \ge 2. \end{cases}$

Solution:

From given

$$\int_0^\infty f(x)\sin sx\;dx = \begin{cases} 1, & 0 \le s < 1 \\ 2, & 1 \le s < 2 \\ 0, & s \ge 2 \end{cases}$$

Now multiply $\sqrt{\frac{2}{\pi}}$ on both sides

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\;dx = \begin{cases} \sqrt{\frac{2}{\pi}}, & 0 \le s < 1 \\ 2\sqrt{\frac{2}{\pi}}, & 1 \le s < 2 \\ 0 & s \ge 2 \end{cases}$$

Since by Fourier sine integral & inverse Fourier sine transform

$$F_s[S] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\;dx \qquad \& \qquad f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s[S]\sin sx\;ds$$

$$\therefore f(x) = \sqrt{\frac{2}{\pi}}\left[\int_0^1 \sqrt{\frac{2}{\pi}}\sin sx\;ds \quad + \int_1^2 2\sqrt{\frac{2}{\pi}}\sin sx\;ds \quad + 0\right]$$

$$= \sqrt{\frac{2}{\pi}} \times \sqrt{\frac{2}{\pi}}\left[\int_0^1 \sin sx\;ds \quad + \int_1^2 2\sin sx\;ds\right]$$

$$= \frac{2}{\pi}\left(\left[-\frac{\cos sx}{x}\right]_0^1 + 2\left[-\frac{\cos sx}{x}\right]_1^2\right)$$

$$= \frac{2}{\pi}\left[\left(-\frac{\cos x}{x} + \frac{\cos 0}{x}\right) + 2\left(-\frac{\cos 2x}{x} + \frac{\cos x}{x}\right)\right]$$

$$= \frac{2}{\pi}\left[-\frac{\cos x}{x} + \frac{1}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right]$$

$$\therefore f(x) = \frac{2}{\pi}\left[\frac{1}{x} + \frac{\cos x}{x} - \frac{2\cos 2x}{x}\right]$$

UNIT-V

PART A

1. Find $Z[f(n)]$, **where** $f(n)=n$ **for** $n=0,1,2,..$

Solution:

W.K.T , Z- transform formula is

$$Z\{x(n)\}=\sum_{n=0}^{\infty}x(n)\,z^{-n}$$

$$Z[n]=\sum_{n=0}^{\infty}n\,z^{-n}\qquad=\sum_{n=0}^{\infty}\frac{n}{z^n}$$

$$Z[n]=0+\frac{1}{z}+\frac{2}{z^2}+\frac{3}{z^3}+\cdots$$

$$=\frac{1}{z}\left[1+2\left(\frac{1}{z}\right)+3\left(\frac{1}{z}\right)^2+\cdots\right]$$

$$=\frac{1}{z}\left[\left(1-\frac{1}{z}\right)^{-2}\right]\qquad\left[\because\ (1-x)^{-2}=1+2x+3x^2+\cdots\right.$$

$$=\frac{1}{z}\left[\left(\frac{z-1}{z}\right)^{-2}\right]$$

$$Z[n]=\frac{1}{z}\left[\frac{z}{z-1}\right]^2=\frac{1}{z}\left[\frac{z^2}{(z-1)^2}\right]$$

$$Z[n]=\frac{z}{(z-1)^2}$$

2. State initial and final value theorem in z **transforms.**

Solution:

Initial value theorem:

$$If\ Z[f_n]=F(z),then\ f(0)=\lim_{z\to\infty}F(z)$$

Final value theorem:

$$\text{If}\ Z[f_n]=F(z),then\ \lim_{n\to\infty}f_n=\lim_{z\to 1}(z-1)F(z)$$

3. Find the Z-transform of $x(n)=\begin{cases}\frac{a^n}{n!} & for\quad n\ge 0\\ 0 & otherwise\end{cases}$

Solution:

We know that $Z\{x(n)\}=\sum_{n=0}^{\infty}x(n)\ z^{-n}$

$$Z\left[\frac{a^n}{n!}\right]=\sum_{n=0}^{\infty}\frac{a^n}{n!}\ z^{-n}$$

$$Z\left[\frac{a^n}{n!}\right]=\sum_{n=0}^{\infty}\frac{a^n}{n!}\left[\frac{1}{z}\right]^n$$

$$=\sum_{n=0}^{\infty}\frac{\left[\frac{a}{z}\right]^n}{n!}$$

$$=1+\frac{\left[\frac{a}{z}\right]^1}{1!}+\frac{\left[\frac{a}{z}\right]^2}{2!}+\frac{\left[\frac{a}{z}\right]^3}{3!}+\cdots$$

$$Z\left[\frac{a^n}{n!}\right]=e^{\frac{a}{z}}\qquad\left(\because\ 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)=e^x$$

4. Form a difference equation by eliminating the arbitrary constants from $u_n = A2^{n+1}$.

Solution:

Given $U_n = A2^{n+1}$ .... (1)

Then $U_{n+1} = A2^{n+2}$

$\Longrightarrow U_{n+1} = A2^{n+1}.2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$

Eliminating A from (1) & (2) we get

$$\Longrightarrow \begin{vmatrix} U_n & 1 \\ U_{n+1} & 2 \end{vmatrix} = 0$$

$$2U_n - U_{n+1} = 0$$

5. Form a difference equation by eliminating the arbitrary constants from $y_n = A3^n$.

Solution:

Given $y_n = A.3^n$ ....(1)

Then $y_{n+1} = A.3^{n+1}$

$\Longrightarrow y_{n+1} = A.3^n(3)$

$\Longrightarrow y_{n+1} = 3A.3^n$ ....(2)

Eliminating A from (1) & (2) we get

$$\Longrightarrow \begin{vmatrix} y_n & 1 \\ y_{n+1} & 3 \end{vmatrix} = 0$$

$$\Longrightarrow 3y_n - y_{n+1} = 0$$

$$\Longrightarrow y_{n+1} - 3y_n = 0$$

6. Find the Z-transform of $\frac{1}{n!}$.

Solution:

We know that $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n)\ z^{-n}$

$$Z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{1}{z}\right]^n$$

$$= 1 + \frac{1}{1!}\left[\frac{1}{z}\right]^1 + \frac{1}{2!}\left[\frac{1}{z}\right]^2 + \frac{1}{3!}\left[\frac{1}{z}\right]^3 + \cdots$$

$$= 1 + \frac{\left[\frac{1}{z}\right]}{1!} + \frac{\left[\frac{1}{z}\right]^2}{2!} + \cdots$$

$$= e^{\frac{1}{z}} \qquad \left(\because\ 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = e^x$$

7. Find the Z transform of $\frac{1}{n}$.

Solution:

W.K.T $Z\{x(n)\} = \sum_{n=0}^{\infty} x(n)\, z^{-n}$

$$Z\left\{\frac{1}{n}\right\} = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{z}\right)^n$$

$$= \sum_{n=1}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n}$$

$$= \frac{\left(\frac{1}{z}\right)^1}{1} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \cdots \ldots$$

$$= -log\left(1 - \frac{1}{z}\right) \qquad \left(\because\ x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$$

$$= -\log(1 - x)$$

$$= -log\left(\frac{z-1}{z}\right)$$

$$= log\left(\frac{z-1}{z}\right)^{-1}$$

$$\therefore\ Z\left\{\frac{1}{n}\right\} = log\left(\frac{z}{z-1}\right)$$

8. Find the Z-transform of a^n.

Solution:

W.K.T Z- transform formula is

$$\mathrm{Z}\ \{\mathrm{x(n)}\} = \sum_{n=0}^{\infty} x(n)\, \mathrm{z}^{-\mathrm{n}}$$

$$\mathrm{Z}\ [a^n] = \sum_{n=0}^{\infty} a^n\, \mathrm{z}^{-\mathrm{n}}$$

$$= \textstyle\sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \cdots$$

$$\mathrm{Z}\ [a^n] = \left[1 - \frac{a}{z}\right]^{-1} \quad [\therefore (1-x)^{-1} = 1 + x + x^2 + \cdots]$$

$$= \left[\frac{z-a}{z}\right]^{-1}$$

$$Z\ [a^n] = \frac{z}{z-a} \quad |z| > |a|$$

9. Solve $y_{n+1} - 2y_n = 0$ given $y_0 = 3$.

Solution:

Given $y_{n+1} - 2y_n = 0$

$$z[y_{n+1} - 2y_n] = 0$$

$$z[y_{n+1}] - 2Z[y_n] = 0$$

$$z[Y(z) - y_0] - 2Y(z) = 0$$

Also given $y_0 = 3$ $\qquad \therefore\ zY(z) - 3z - 2Y(z) = 0$

$$Y(z)(z-2) = 3z$$

$$Y(z) = \frac{3z}{(z-2)}$$

$$Z(y(n)) = \frac{3z}{(z-2)} \qquad [\ \because Y(z) = Z(y(n))$$

$$= Z^{-1}\left[\frac{3z}{(z-2)}\right] \quad = 3Z^{-1}\left[\frac{z}{(z-2)}\right]$$

$y(n) = 3.2^n \qquad [\ \because a^n = Z^{-1}\left[\frac{z}{z-a}\right]$

10. State the Convolution theorem on Z transforms.

Solution:

If $Z[f(n)] = F(z)$ & $Z[g(n)] = G(z)$ then $Z[f(n) * g(n)] = F(z)G(z)$

11. Obtain $Z^{-1}\left[\frac{z}{(z+1)(z+2)}\right]$

Solution:

Let $X(Z) = \frac{z}{(z+1)(z+2)}$

$$\frac{X(Z)}{z} = \frac{1}{(z+1)(z+2)}$$

$$\frac{1}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$

$$1 = A(z+2) + B(z+1)$$

Put $z = -1$ we get, $1 = A(-1+2) + B(0)$

$$\boldsymbol{A = 1}$$

Put z = -2 we get, $1 = A(0) + B(-2+1)$

$$\boldsymbol{B = -1}$$

$$\frac{X(Z)}{z} = \frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2}$$

$$X(Z) = \frac{z}{z+1} - \frac{z}{z+2}$$

$$Z\{x(n)\} = \frac{z}{z+1} - \frac{z}{z+2}$$

$$x(n) = Z^{-1}\left[\frac{z}{z+1}\right] - Z^{-1}\left[\frac{z}{z+2}\right]$$

$$x(n) = (-1)^n - (-2)^n$$

12. If $Z(x(n)) = X(z)$ **then show that** $Z(a^n x(n)) = X\left(\frac{z}{a}\right)$.

Solution:

By definition of z-transform

$$X(z) = z[x(n)] = \sum_{n=0}^{\infty} x(n)z^{-n}$$

$$Z[anx(n)] = \sum_{n=0}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} x(n)\left[\frac{z}{a}\right]^{-n}$$

$$= X\left[\frac{z}{a}\right]$$

13. Find the Z **-transform of** $\sin\frac{n\pi}{2}$ **.**

Solution:

We know that $z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$

Put $\theta = \frac{\pi}{2}$ we get

$$z\left[\sin\frac{n\pi}{2}\right] = \frac{z\sin\frac{\pi}{2}}{z^2 - 2z\cos\frac{\pi}{2} + 1}$$

$$= \frac{z\,(1)}{z^2 - 2z(0) + 1}$$

$$= \frac{z}{z^2 + 1}$$

14. Find the difference equation generated by $y_n = an + b2^n$

Solution:

Given $y_n = b2^n + an$ ....(1)

Then $y_{n+1} = b2^{n+1} + a(n+1)$

$\Rightarrow y_{n+1} = b2^n(2) + a(n+1)$

$\Rightarrow y_{n+1} = 2b2^n + a(n+1)$ ....(2)

Then $y_{n+2} = b2^{n+2} + a(n+2)$

$\Rightarrow y_{n+2} = b2^n(2^2) + a(n+2)$

$\Rightarrow y_{n+2} = 4b2^n + a(n+2)$ ....(3)

Eliminating a and b from (1), (2) & (3) we get

$$\begin{bmatrix} y_n & 1 & n \\ y_{n+1} & 2 & n+1 \\ y_{n+2} & 4 & n+2 \end{bmatrix} = 0$$

$$y_n[2(n+2) - 4(n+1)] - y_{n+1}[n+2-4n] + y_{n+2}[n+1-2n] = 0$$

$$y_n[2n+4-4n-4] - y_{n+1}[-3n+2] + y_{n+2}[-n+1] = 0$$

$$y_n[-2n] - y_{n+1}[-3n+2] - y_{n+2}[n-1] = 0$$

$$\Rightarrow 2ny_n + (2-3n)y_{n+1} + (n-1)y_{n+2} = 0$$

15. Define the unit step sequence. Write its Z transform

Solution:

The unit step sequence u(n) has values

$$u(n) = \begin{cases} 1 & for\ \ n > 0 \\ 0 & for\ \ n < 0 \end{cases}$$

W.K.T

$$Z\{x(n)\} = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$Z\{u(n)\} = \sum_{n=0}^{\infty} u(n)\, z^{-n}$$

$$Z\{u(n)\} = \sum_{n=0}^{\infty} z^{-n} \quad by\ defn.of\ u(n)$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$$

$$= \left[1 - \frac{1}{z}\right]^{-1}$$

$$Z\{u(n)\} = \left[\frac{z-1}{z}\right]^{-1}$$

$$Z\{u(n)\} = \frac{z}{z-1}$$

PART-B

16. Find the Z-transform of $\frac{1}{n(n+1)}, n \geq 1$

Solution:

By partial fraction

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{(n+1)} \ldots\ldots\ldots\ldots\ldots (1)$$

$$1 = A(n+1) + Bn$$

Let $n = 0 \Rightarrow \quad 1 = A(0+1) + 0$

$$\boldsymbol{A = 1}$$

Let $n = -1 \Rightarrow \quad 1 = A(0) + B(-1)$

$$\boldsymbol{B = -1}$$

Sub A=1 , B=-1 in equation [1]

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{(n+1)}$$

Taking z-transform on both sides

$$z\left[\frac{1}{n(n+1)}\right] = z\left[\frac{1}{n}\right] - z\left[\frac{1}{(n+1)}\right]$$

$$= \log\frac{z}{z-1} - z\log\frac{z}{z-1}$$

$$= (1-z)\log\frac{z}{z-1}$$

17. Find the Z **-transform of** $\sin n\theta$ & $\cos n\theta$ **. Hence deduce the** Z **-transforms of** $a^n \sin n\theta$ & $\cos(n+1)\theta$

Solution:

To find: $z[\cos n\theta]$ and $z[\sin n\theta]$

We know that

$$z[a^n] = \frac{z}{z-a}, |z| > |a|$$

$$z[(e^{i\theta})^n] = \frac{z}{z-e^{i\theta}}$$

$$z[e^{in\theta}] = \frac{z}{z-(\cos\theta + i\sin\theta)}$$

$$z[\cos n\theta + i\sin n\theta] = \frac{z}{(z-\cos\theta) - i\sin\theta}$$

$$z[\cos n\theta] + iz[\sin n\theta] = \frac{z}{(z-\cos\theta) - i\sin\theta}$$

$$z[\cos n\theta] + iz[\sin n\theta] = \left[\frac{z}{(z-\cos\theta) - i\sin\theta}\right]\left[\frac{(z-\cos\theta) + i\sin\theta}{(z-\cos\theta) + i\sin\theta}\right]$$

$$= \frac{z(z-\cos\theta) + i\,z\sin\theta}{z^2 - 2z\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{z(z-\cos\theta) + i\,z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$\Longrightarrow z[\cos n\theta] + iz[\sin n\theta] = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} + i\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Equating the real and imaginary parts, we get

$$z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$

$$z[\sin n\theta] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

(i) To find $z[\cos(n+1)\theta]$

By shifting theorem,

$$z[\cos(n+1)\theta] = z[z[\cos n\theta]-\cos 0]$$

$$= z\left[\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}-1\right]$$

$$= z\left[\frac{z^2-z\cos\theta-z^2+2z\cos\theta-1}{z^2-2z\cos\theta+1}\right]$$

$$= z\left[\frac{z\cos\theta-1}{z^2-2z\cos\theta+1}\right]$$

$$= \frac{z^2\cos\theta-z}{z^2-2z\cos\theta+1}$$

(ii) To find $z[a^n \sin n\theta]$

We know that $z[a^n f(n)] = F\left[\frac{z}{a}\right]$

$$z[a^n\sin n\theta] = [z[\sin n\theta]]_{z\to\frac{z}{a}}$$

$$= \left[\frac{z\sin\theta}{z^2-2z\cos\theta+1}\right]_{z\to\frac{z}{a}}$$

$$= \frac{\left(\frac{z}{a}\right)\sin\theta}{\left(\frac{z}{a}\right)^2-2\left(\frac{z}{a}\right)\cos\theta+1}$$

$$= \frac{\left(\frac{z\sin\theta}{a}\right)}{\frac{z^2-2az\cos\theta+a^2}{a^2}}$$

$$= \frac{az\sin\theta}{z^2-2az\cos\theta+a^2}$$

18. Find $Z(na^n\sin n\theta)$**.**

Solution:

We know that

$$z[nf(n)] = -z\frac{d}{dz}[z[f(n)]]$$

$$z[na^n\sin n\theta] = -z\frac{d}{dz}[z[a^n\sin n\theta]]$$

$$= -z\frac{d}{dz}\left[\frac{az\sin\theta}{z^2-2az\cos\theta+a^2}\right] \qquad \left[\because d\left(\frac{u}{v}\right) = \frac{vdu-udv}{v^2}\right]$$

$$= -z\left[\frac{(z^2-2az\cos\theta+a^2)(a\sin\theta)-(az\sin\theta)(2z-2a\cos\theta)}{(z^2-2az\cos\theta+a^2)^2}\right]$$

$$= -z\left[\frac{(z^2a\sin\theta-2a^2z\sin\theta\cos\theta+a^3\sin\theta-2az^2\sin\theta+2a^2z\sin\theta\cos\theta)}{(z^2-2az\cos\theta+a^2)^2}\right]$$

$$= \frac{-z(-z^2a\sin\theta+a^3\sin\theta)}{(z^2-2az\cos\theta+a^2)^2}$$

$$= \frac{za([(z^2-a^2)\sin\theta]}{(z^2-2az\cos\theta+a^2)^2}$$

19. Solve $Z(\cos n\theta)$ and hence deduce $Z\left(\cos\frac{n\pi}{2}\right)$.

Solution:

$$z[a^n] = \frac{z}{z-a}, |z| > |a|$$

$$z[(e^{i\theta})^n] = \frac{z}{z-e^{i\theta}}$$

$$z[e^{in\theta}] = \frac{z}{z-(\cos\theta + i\sin\theta)}$$

$$z[\cos n\theta + i\sin n\theta] = \frac{z}{(z-\cos\theta) - i\sin\theta}$$

$$z[\cos n\theta] + iz[\sin n\theta] = \frac{z}{(z-\cos\theta) - i\sin\theta}$$

$$z[\cos n\theta] + iz[\sin n\theta] = \left[\frac{z}{(z-\cos\theta) - i\sin\theta}\right]\left[\frac{(z-\cos\theta) + i\sin\theta}{(z-\cos\theta) + i\sin\theta}\right]$$

$$= \frac{z(z-\cos\theta) + i\,z\sin\theta}{z^2 - 2z\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{z(z-\cos\theta) + i\,z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$\Rightarrow z[\cos n\theta] + iz[\sin n\theta] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} + i\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

Equating the real and imaginary parts, we get

$$z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$

We know that $z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$

Put $\theta = \frac{\pi}{2}$ we get

$$z[\cos\frac{n\pi}{2}] = \frac{z(z-\cos\frac{\pi}{2})}{z^2-2z\cos\frac{\pi}{2}+1}$$

$$= \frac{z^2}{z^2+1} \qquad [\because \cos\frac{\pi}{2} = 0]$$

20. Find the inverse Z-transform of $\frac{10z}{z^2-3z+2}$.

Solution:

Given

$$Z^{-1}\left[\frac{10z}{z^2-3z+2}\right] = Z^{-1}\left[\frac{10z}{(z-1)(z-2)}\right]$$

Let $X(z) = \frac{10z}{(z-1)(z-2)}$

$$\frac{X(z)}{z} = \frac{10}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \qquad \ldots(1)$$

$$10 = A(z-2) + B(z-1)$$

Let $z = 1$ $\quad 10 = -A \Rightarrow \boldsymbol{A = -10}$

Let $z = 2$ $\qquad 10 = 0 + B$

$B = 10$

$$\frac{X(z)}{z} = \frac{-10}{z-1} + \frac{10}{z-2}$$

$$X(z) = \frac{-10z}{z-1} + \frac{10z}{z-2}$$

Taking Z^{-1} on both sides

$$Z^{-1}(X(z)) = -10Z^{-1}\left[\frac{z}{z-1}\right] + 10Z^{-1}\left[\frac{z}{z-2}\right]$$

$$Z^{-1}\left[\frac{10z}{z^2-3z+2}\right] = -10(1)^n + 102^n$$

$$\therefore\ Z^{-1}\left[\frac{10z}{z^2-3z+2}\right] = 10(\,2^n - 1)$$

21. Find the inverse Z-transform of $\dfrac{z^3-20z}{(z-2)^3(z-4)}$

Solution:

Let $X(z) = \frac{z^3-20z}{(z-2)^3(z-4)}$

$$\frac{X(z)}{z} = \frac{z^2-20}{(z-2)^3(z-4)}$$

$$\frac{X(z)}{z} = \frac{z^2-20}{(z-2)^3(z-4)} = \frac{A}{z-2} + \frac{B}{(z-2)^2} + \frac{C}{(z-2)^3} + \frac{D}{z-4}\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$z^2 - 20 = A(z-2)^2(z-4) + B(z-2)(z-4) + D(z-2)^3 + c(z-4)$$

Put $z = 2$ we get

$$4 - 20 = C(-2)$$

$$-16 = -2C$$

$$\boldsymbol{C = 8}$$

Put $z = 4$ we get

$$16 - 20 = D(8)$$

$$-4 = 8D$$

$$\boldsymbol{D = -1/2}$$

Equating the coefficients of z^3 on both sides, we get

$$0 = A + D$$

$$A = -D$$

$$\boldsymbol{A = -1/2}$$

Put $z = 0$ we get

$$-20 = -16A + 8B - 4C - 8D$$

$$8B = -20 + 16A + 4C + 8D$$

$$8B = -20 + 16(-1/2) + 4(8) + 8(-1/2)$$

$$8B = -20 + 8 + 32 - 4$$

$$8B = 40 - 24$$

$8B = 16$

$\boldsymbol{B = 2}$

Substituting these values in equation (1), we get

$$\frac{X(z)}{z} = \frac{z^2 - 20}{(z-2)^3(z-4)} = \frac{1/2}{z-2} + \frac{2}{(z-2)^2} + \frac{8}{(z-2)^3} - \frac{1/2}{z-4}$$

$$X(z) = \frac{1}{2}\frac{z}{z-2} + \frac{2z}{(z-2)^2} + \frac{8z}{(z-2)^3} - \frac{1}{2}\frac{z}{z-4}$$

Taking z^{-1} on both sides we get

$$z^{-1}[X(z)]\frac{1}{2}z^{-1}[\frac{z}{z-2}] + 2z^{-1}[\frac{z}{(z-2)^2}] + 8z^{-1}[\frac{z}{(z-2)^3}] - \frac{1}{2}[\frac{z}{z-4}]$$

$$\Rightarrow z^{-1}[X(z)] = \frac{1}{2}2^n + n2^n + 2^n n(n-1) - \frac{1}{2}4^n$$

$$\Rightarrow z^{-1}[X(z)] = 2^n[\frac{1}{2} + n + n(n-1)] - \frac{1}{2}4^n$$

$$\Rightarrow z^{-1}[X(z)] = 2^n[\frac{1}{2} + n + n^2 - n] - \frac{1}{2}4^n$$

$$\Rightarrow z^{-1}[X(z)] = 2^n[\frac{1}{2} + n^2] - \frac{1}{2}4^n$$

$$\Rightarrow z^{-1}[\frac{z^3 - 20z}{(z-2)^3(z-4)}] = 2^{n-1} + n^2 2^n - \frac{1}{2}4^n$$

22. Using convolution theorem find the inverse Z-transform of $\frac{8z^2}{(2z-1)(4z+1)}$.

Solution:

Given

$$Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right] = Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\cdot\frac{z}{z-\frac{1}{4}}\right]$$

$$Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right] = Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] * Z^{-1}\left[\frac{z}{z-\frac{1}{4}}\right]$$

$$= \left(\frac{1}{2}\right)^n * \left(\frac{1}{4}\right)^n$$

$$Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right] = \sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(\frac{1}{4}\right)^{n-k}$$

$$= \left(\frac{1}{4}\right)^n \sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(\frac{1}{4}\right)^{-k}$$

$$= \left(\frac{1}{4}\right)^n \sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(\frac{1}{2^2}\right)^{-k}$$

$$= \left(\frac{1}{4}\right)^n \sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{-2k}$$

$$Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right] = \left(\frac{1}{4}\right)^n \sum_{k=0}^{n}\left(\frac{1}{2}\right)^{-k} = \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} 2^k$$

$$= \left(\frac{1}{4}\right)^n [1 + (2) + (2)^2 + \cdots + (2)^n]$$

$$since\ 1 + r + r^2 + r^3 + \cdots + r^n. = \ = \frac{r^{n+1}-1}{r-1} \quad for\ r < 1$$

$$= \left(\frac{1}{2}\right)^{2n}\left[\frac{(2)^{n+1}-1}{2-1}\right] \qquad by\ G.P$$

$$Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right] = \left(\frac{1}{2}\right)^{2n}[(2)^{n+1}-1] \quad = \left(\frac{1}{2}\right)^{2n}\left[\frac{1}{2^{-n-1}} - 1\right]$$

$$= \left(\frac{1}{2}\right)^{2n}\left[\left(\frac{1}{2}\right)^{-n-1} - 1\right]$$

$$Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right] = \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{2}\right)^{2n}$$

23. Using convolution theorem, find the inverse Z - transform of $\left(\frac{z^2}{(z+a)^2}\right)$.

Solution:

$$Z^{-1}\left[\frac{z^2}{(z+a)^2}\right] = Z^{-1}\left[\frac{z}{z+a}\ \frac{z}{z+a}\right]$$

$$= Z^{-1}\left[\frac{z}{z+a}\right] * Z^{-1}\left[\frac{Z}{z+a}\right]$$

$$= (-a)^n * (-a)^n$$

$$= \sum_{k=0}^{n} (-a)^k (-a)^{n-k}$$

$$= (-a)^n \sum_{k=0}^{n} (1)^k$$

$$= (-a)^n [1 + 1 + 1 + \cdots]$$

$$Z^{-1}\left[\frac{z^2}{(z+a)^2}\right] = (\mathrm{n}+1)\,(-a)^n$$

24. By using convolution theorem, prove that the inverse Z-transform of $\frac{z^2}{(z-a)(z-b)}$.

Solution:

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\frac{z}{z-a}\ \frac{z}{z-b}\right]$$

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\frac{z}{(z-a)}\right] * Z^{-1}\left[\frac{Z}{z-b}\right]$$

$$= a^n * b^n$$

$$= \sum_{k=0}^{n} a^k b^{n-k}$$

$$= b^n \sum_{k=0}^{n} \left[\frac{a}{b}\right]^k$$

$$= b^n \left[1 + \frac{a}{b} + \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^3 + \cdots + \left(\frac{a}{b}\right)^n .\right]$$

$$since\ 1 + r + r^2 + r^3 + \cdots + r^n = \ = \frac{r^{n+1} - 1}{r - 1}$$

$$\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\frac{a}{b} - 1} = b^n \left[\frac{\frac{a^{n+1} - b^{n+1}}{b^{n+1}}}{\frac{a-b}{b}}\right]$$

$$= b^n \left[\frac{\frac{a^{n+1} - b^{n+1}}{b^n\, b}}{\frac{a-b}{b}}\right]$$

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{a^{n+1} - b^{n+1}}{a - b}$$

25. Form a difference equation from the relation $y_n = a + b3^n$.

Solution:

26. Construct the difference equation from $y_n = (A + Bn)(-3)^n$

Solution:

Given $y_n = (A + Bn)(-3)^n$

$$= A(-3)^n + Bn(-3)^n$$

$$y_{n+1} = (A + B(n+1))(-3)^{n+1}$$

$$= A(-3)^n(-3) - 3B(n+1)(-3)^n(-3)$$

$$= -3A(-3)^n - 3B(n+1)(-3)^n$$

$$y_{n+2} = (A + B(n+2))(-3)^{n+2}$$

$$= A(-3)^{n+2} + B(n+2)(-3)^{n+2}$$

$$= 9A(-3)^n + 9B(n+2)(-3)^n$$

Eliminating A and B we get

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & -3 & -3(n+1) \\ y_{n+2} & 9 & 9(n+2) \end{vmatrix} = 0$$

$$y_n[-27(n+2) + 27(n+1)] - y_{n+1}[9(n+2) - 9n] + y_{n+2}[-3(n+1) + 3n] = 0$$

$$y_n[-27n - 54 + 27n + 27] - y_{n+1}[9n + 18 - 9n] + y_{n+2}[-3n - 3 + 3n] = 0$$

$$y_n(-27) - y_{n+1}(18) + y_{n+2}(-3) = 0$$

$$-27y_n - 18y_{n+1} - 3y_{n+2} = 0$$

$$(\div\ -3) \qquad 9y_n + 6y_{n+1} + y_{n+2} = 0$$

27. Using Z-transform solve $y_{(n+2)} - 3y_{(n+1)} - 10y_{(n)} = 0, y_0 = 1 \& y_1 = 0$.

Solution:

Given $y_{n+2} - 3y_{n+1} - 10y_n = 0, y_0 = 1 \ \& \ y_1 = 0$

Taking z-transform on both sides, we get

Using the formula $Z[y_n] = Y(z)$

$$Z[y_{n+1}] = zY(z) - zy_0$$

$$Z[y_{n+2}] = z^2Y(z) - z^2y_0 - zy_1$$

$$z[y_{n+2}]-3z[y_{n+1}]-10z[y_n]=0, y_0=1 \;\&\; y_1=0$$

$$[z^2Y(z)-z^2y_0-zy_1]-3[zY(z)-zy_0]-10Y(z)=0$$

$$[z^2Y(z)-z^2]-3[zY(z)-z]-10Y(z)=0$$

$$z^2Y(z)-z^2-3zY(z)+3z-10Y(z)=0$$

$$[z^2-3z-10]Y(z)-z^2+3z=0$$

$$[z^2-3z-10]Y(z)=z^2-3z$$

$$Y(z)=\frac{z^2-3z}{[z^2-3z-10]}=\frac{z(z-3)}{(z-5)(z+2)}$$

To apply residue method multiply both sides by z^{n-1}

$$Y(z)z^{n-1}=\frac{z(z-3)}{(z-5)(Z+2)}z^{n-1}=\frac{(z-3)}{(z-5)(z+2)}z^n$$

Here $z=5 \;\&\; z=-2$ is a pole of order 1

$$R_1=\operatorname*{Res}_{z=5} Y(z)z^{n-1}=\lim_{z\to 5}(z-5)\frac{(z-3)}{(z-5)(z+2)}z^n$$

$$=\lim_{z\to 5}\frac{(z-3)}{(z+2)}z^n=\frac{(5-3)}{(5+2)}5^n$$

$$R_1=\frac{2}{7}5^n$$

$$R_2=\operatorname*{Res}_{z=-2} Y(z)z^{n-1}=\lim_{z\to -2}(z+2)\frac{(z-3)}{(z-5)(Z+2)}z^n$$

$$=\lim_{z\to -2}\frac{(z-3)}{(z-5)}z^n$$

$$=\frac{(-2-3)}{(-2-5)}(-2)^n=\frac{-5}{-7}(-2)^n$$

$$\therefore\quad R_2=\frac{5}{7}(-2)^n$$

By Cauchy' s Residue theorem $y(n)=Sum\ of\ the\ residues=R_1+R_2$

$$y(n)=\frac{2}{7}5^n+\frac{5}{7}(-2)^n$$

28. Solve $y_{(n+2)}+4y_{(n+1)}+3y_{(n)}=2^n$ **with** $y_0=0 \;\&\; y_1=1$ **using Z-transform.**

Solution:

Using the formula

$$Z[y_n]=Y(z)$$

$$Z[y_{n+1}]=zY(z)-zy_0$$

$$Z[y_{n+2}]=z^2Y(z)-z^2y_0-zy_1$$

Given $y_{n+2}+4y_{n+1}+3y_n=2^n$

$$z[y_{n+2}]+4z[y_{n+1}]+3z[y_n]=z[2^n]$$

$$z^2Y(z)-z^2y_0-zy_1+4[zY(z)-zy_0]+3Y(z)=\frac{z}{z-2}$$

Given $y_0=0; y_1=1$

$$z^2Y(z)-z+4zY(z)+3Y(z)=\frac{z}{z-2}$$

$$[z^2+4z+3]Y(z)=\frac{z}{z-2}+z$$

$$[z^2+4z+3]Y(z)=\frac{z+z(z-2)}{z-2}$$

$$[z^2+4z+3]Y(z)=\frac{z+z^2-2z}{z-2}$$

$$(z+1)(z+3)Y(z)=\frac{z^2-z}{z-2}$$

$$Y(z)=\frac{z^2-z}{(z+1)(z+3)(z-2)}$$

To apply residue method multiply both sides by z^{n-1}

$$Y(z)z^{n-1}=\frac{z^n(z-1)}{(z+1)(z+3)(z-2)}$$

$Y(z)z^{n-1}$ has a simple pole at $2, -1$ *and* -3

$z=2$ is a simple pole

$z=-1$ is a simple pole

$z=-3$ is a simple pole

(i) $R_1=\operatorname{Res}_{z=-2} Y(z)z^{n-1}=\lim_{z\to 2}(z-2)\frac{z^n(z-1)}{(z+1)(z+3)(z-2)}$

$$=\lim_{z\to 2}\frac{z^n(z-1)}{(z+1)(z+3)}$$

$$=\frac{(2)^n}{(3)(5)}$$

$$R_1=\frac{2^n}{15}$$

(ii) $R_2=\operatorname{Res}_{z=-1} Y(z)z^{n-1}=\lim_{z\to -1}(z+1)\frac{z^n(z-1)}{(z+1)(z+3)(z-2)}$

$$=\lim_{z\to -1}\frac{z^n(z-1)}{(z-2)(z+3)}$$

$$=\frac{(-1)^n(-2)}{(-3)(2)}$$

$$R_2=\frac{(-1)^n}{3}$$

(iii) $R_3=\operatorname{Res}_{z=-3} Y(z)z^{n-1}=\lim_{z\to -3}(z+3)\frac{z^n(z-1)}{(z+1)(z+3)(z-2)}$

$$=\lim_{z\to -3}\frac{z^n(z-1)}{(z-2)(z+1)}$$

$$=\frac{(-3)^n(-4)}{(-5)(-2)}$$

$$=\frac{2(-3)^n}{(-5)}$$

$$R_3=\frac{3^n}{5}2(-1)^{n+1}$$

By Cauchy' s Residue theorem

$$y(n) = Sum\ of\ the\ residues = R_1 + R_2 + R_3$$

$$= \frac{2^n}{15} + \frac{(-1)^n}{3} + \frac{3^n}{5} 2(-1)^{n+1}$$

$$= \frac{1}{15} 2^n + \frac{1}{3}(-1)^n + \frac{2}{5}(-1)^{n+1} 3^n$$

$$y(n) = \frac{1}{15} 2^n + \frac{1}{3}(-1)^n + \frac{2}{5}(-1)^{n+1} 3^n$$

29. Solve the equation $u_{n+2} + 6u_{n+1} + 9u_n = 2^n$ **given** $u_0 = u_1 = 0$.

Solution:

Using the formula

$$Z[y_n] = Y(z)$$

$$Z[y_{n+1}] = zY(z) - zy_0$$

$$Z[y_{n+2}] = z^2Y(z) - z^2y_0 - zy_1$$

Given $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$

$$Z[y_{n+2}] + 6Z[y_{n+1}] + 9Z[y_n] = Z[2^n]$$

$$z^2Y(z) - z^2y_0 - zy_1 + 6[zY(z) - zy_0] + 9Y(z) = \frac{z}{z-2}$$

$$z^2Y(z) + 6zY(z) + 9Y(z) = \frac{z}{z-2}$$

$$[z^2 + 6z + 9]Y(z) = \frac{z}{z-2}$$

$$(z+3)^2Y(z) = \frac{z}{z-2}$$

$$Y(z) = \frac{z}{(z-2)(z+3)^2}$$

To apply residue method multiply both sides by z^{n-1}

$$Y(z)z^{n-1} = \frac{z^n}{(z-2)(z+3)^2}$$

$Z = 2$ is a simple pole, $Z = -3$ is a pole of order 2

$$R_1 = \operatorname*{Res}_{z=2} Y(z)z^{n-1} = \lim_{z\to 2}(z-2)\frac{z^n}{(z-2)(z+3)^2}$$

$$= \lim_{z\to 2}\frac{z^n}{(z+3)^2} = \frac{2^n}{25}$$

$$R_2 = \operatorname*{Res}_{z=-3} Y(z)z^{n-1} = \lim_{z\to -3}\frac{1}{1!}\frac{d}{dz}(z+3)^2 \frac{z^n}{(z-2)(z+3)^2}$$

$$= \lim_{z\to -3}\frac{d}{dz}\left(\frac{z^n}{z-2}\right)$$

$$= \lim_{z\to -3}\left[\frac{(z-2)nz^{n-1} - z^n}{(z-2)^2}\right]$$

$$= \frac{(-5)^n(-3)^{n-1} - (-3)^n}{(-5)^2}$$

$$\operatorname*{Res}_{z=-3} Y(z)z^{n-1} = \frac{(-3)^n[-5n(-3)^{-1} - 1]}{25}$$

$$= \frac{(-3)^n\left[\frac{5}{3}n - 1\right]}{25}$$

$$\underset{z=-3}{\text{Res}}\, Y(z)z^{n-1} = \frac{(-3)^n[5n-3]}{75}$$

By Cauchy' s Residue theo

$$y(n) = Sum\ of\ the\ residues = R_1 + R_2$$

$$= \frac{2^n}{25} + \frac{(-3)^n[5n-3]}{75}$$

$$y(n) \quad = \frac{1}{25}\left[2^n - (-3)^n + \frac{5}{3}n(-3)^n\right]$$