20MA4T4-NUMERICAL METHODS

FOR II - B.E. (EEE&CIVIL) / IV SEMESTER

SYLLABUS

Semester	Programme	Course Code	Course Name	L	T	P	C
IV	Common to B.E. EEE & CIVIL Programmes	20MA4T4	NUMERICAL METHODS	3	1	0	4

COURSE LEARNING OUTCOMES (COs)			
After Successful completion of the course, the students should be able to		RBT Level	Topics Covered
CO1	Identify and apply various numerical techniques for solving non-linear equations and systems of linear equations.	K3	1
CO2	Analyse and apply the knowledge of interpolation and determine the integration and differentiation of the functions by using the numerical data.	K4	3
CO3	Categorize various types of interpolation with equal and unequal intervals and apply the concept of cubic spline, approximation of derivatives using interpolation polynomials.	K4	2
CO4	Determine the dynamic behaviour of the system through solution of ordinary differential equations by using numerical methods.	K5	4
CO5	Solve PDE models representing spatial and temporal variations in physical systems through numerical methods.	K3	5

PRE-REQUISITE	Engineering Mathematics I & Engineering Mathematics II

CO / PO MAPPING (1 – Weak, 2 – Medium, 3 – Strong)														
COs	Programme Learning Outcomes (POs)												PSOs	
	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12	PSO1	PSO2
CO1	3	3		3					2	3				
CO2	3	3		3					2	3				
CO3	3	3		3					2	3				
CO4	3	3		3					2	3				
CO5	3	3		3					2	3				

COURSE ASSESSMENT METHODS		
DIRECT	1	Continuous Assessment Tests
	2	Assignments and Tutorials
	3	End Semester Examinations
INDIRECT	1	Course End Survey

COURSE CONTENT		
Topic - 1	**SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS**	**9 + 3**
Solution of algebraic and transcendental equations - Fixed point iteration method – Newton Raphson method - Solution of linear system of equations - Gauss elimination method – Pivoting - Gauss Jordan method – Iterative methods of Gauss Jacobi and Gauss Seidel - Eigenvalues of a matrix by Power method.		
Topic - 2	**INTERPOLATION AND APPROXIMATION**	**9 + 3**
Interpolation with unequal intervals - Lagrange's interpolation – Newton’s divided difference interpolation – Cubic Splines - Interpolation with equal intervals - Newton’s forward and backward difference formulae.		
Topic - 3	**NUMERICAL DIFFERENTIATION AND INTEGRATION**	**9 + 3**
Approximation of derivatives using interpolation polynomials - Numerical integration using Trapezoidal, Simpson’s 1/3 rule – Romberg’s Method - Two point Gaussian quadrature formulae – Evaluation of double integrals by Trapezoidal and Simpson’s 1/3 rules.		
Topic - 4	**INITIAL VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS**	**9 + 3**
Single step methods - Taylor’s series method - Euler’s method - Modified Euler’s method - Fourth order Runge - Kutta method for solving first order equations - Multi step methods - Milne’s predictor corrector methods for solving first order equations.		

Topic - 5	BOUNDARY VALUE PROBLEMS IN ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS	9 + 3

Finite difference methods for solving second order two - point linear boundary value problems - Finite difference techniques for the solution of two dimensional Laplace's and Poisson's equations on rectangular domain – One dimensional heat flow equation by explicit and implicit (Crank Nicholson) method.

THEORY	45		TUTORIAL	15		PRACTICAL	0		TOTAL	60

BOOK REFERENCES	
1	Gerald. C. F. and Wheatley. P. O., **"Applied Numerical Analysis"**, Pearson Education,Asia, 7th Edition, New Delhi, 2006.
2	Grewal, B.S., and Grewal, J.S., **"Numerical Methods in Engineering and Science"**, Khanna Publishers, 9th Edition, New Delhi, 2010
3	Stevan C Chapra, **"Applied Numerical Methods with MAT LAB for Engineers andScientist",**Tata McGraw Hill Publishing Company Limited, 2nd Edition, 2007.
4	P.B Pasil, N P Varma.,**"Numerical Computational Methods"**, Narosa Publishing House 2009
5	Burden, R.L and Faires, J.D, **"Numerical Analysis"**, 9th Edition, Cengage Learning, 2016.

OTHER REFERENCES	
1	**https://www.sobtell.com/blog/38-real-life-applications-of-numerical-analysis**
2	**https://www.scienceabc.com/eyeopeners/why-do-we-need-numerical-analysis-in-everyday-life.html**
3	**https://leverageedu.com/blog/application-of-statistics/**

UNIT-I

SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS

PART – A

FIXED POINT ITERATION METHOD

1. **What is the order of convergence and the condition for convergence of fixed point iteration method?**

Sol:

Order of convergence: 1

Condition for convergence: $|\phi'(x)| < 1$

NEWTON'S METHOD (OR) NEWTON RAPHSON METHOD

2. State the order of convergence and condition for convergence of Newton-Raphson method. (OR)

Write the convergence condition and order of convergence for Newton-Raphson method.

Solution: Order of convergence is two.

Condition for convergence is $|f(x).f''(x)| < |f'(x)|^2$

3. **Find the smallest positive roots of the equation $x^3 - 2x + 0.5 = 0$**

Solution:

f(x) = $x^3 - 2x + 0.5$

$f'(x) = 3x^2 - 2$

f(0) = $0.5(+ve)$

f(x) = $-0.5\ (-ve)$

Hence the roots lies between 0 and 1. Since the value of f(x) at x=0 is very close to zero than the value of f(x) at x=1, we can say that the root is very close to 0. There fore we can assume that $x_0 = 0$. is the initial approximation to the root.

Newton's formula is

$$\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Putting n=0 in (1), we get the first approximation x_1 to the root, given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{0.5}{-2}$$

$$x_1 = 0.25$$

Putting n=1 in (1), we get the second approximation x_2 to the root, given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.25 - \frac{(0.25)^3 - 2(0.25) + 0.5}{3(0.25)^2 - 2}$$

$$= 0.25 - \frac{0.0156}{-1.8125} = 0.2586$$

$$\boldsymbol{x_2 = 0.2586}$$

Putting n=2 in (1), we get the third approximation x_3 to the root, given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.2586 - \frac{(0.2586)^3 - 2(0.2586) + 0.5}{3(0.2586)^2 - 2}$$

$$\boldsymbol{x_3 = 0.2586}$$

Hence the smallest positive root is $\mathbf{0.2586.}$

4. Derive the formula to find the value of $^{1}/_{N}$ where $N \neq 0$, using Newton Raphson method.

Solution:

Let $x = \frac{1}{N}$

$$N = \frac{1}{x}$$

$$\frac{1}{x} - N = 0$$

$$f(x) = \frac{1}{x} - N \ ; \ f'(x) = -\frac{1}{x^2}$$

The Newton's formula is $\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$

$$= x_n - \frac{\left(\frac{1}{x_n} - N\right)}{\left(-\frac{1}{x_n{}^2}\right)}$$

$$= x_n + \left(\frac{1}{x_n} - N\right) X \ \ x_n{}^2$$

$$= x_n + x_n - x_n{}^2 N$$

$$= x_n(2 - N x_n)$$

5. Arrive a formula to find the value of $\sqrt[3]{N}$ where $N \neq 0$, using Newton-Raphson method.

Solution:

$$\text{Let } x = \sqrt[3]{N}$$

$$x^3 = N$$

$$x^3 - N = 0$$

$$f(x) = x^3 - N \quad ; \quad f'(x) = 3x^2$$

By Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n{}^3 - N}{3x_n{}^2} = \frac{3x_n{}^3 - x_n{}^3 + N}{3x_n{}^2}$$

$$= \frac{1}{3}\left[\frac{2x_n{}^3 + N}{x_n{}^2}\right]$$

$$= \frac{1}{3}\left[2x_n + \frac{N}{x_n{}^2}\right], n = 0,1,2, \dots \dots$$

GAUSSIAN ELIMINATION AND GAUSS – JORDON METHODS

6. Give two direct methods to solve a system of linear equation.

Solution: **(AU M/J 2009)**

* Gauss Elimination Method

* Gauss Jordon Method.

7. Compare Gauss – Jacobi and Gauss – Sedial method.

Solution:

S.No	Gauss – Jacobi method	Gauss – Sedial method
1.	Convergence rate is slow	The rate of convergence of Gauss – Sedial method is roughly twice that of Gauss – Jacobi
2.	Indirect method	Indirect method
3.	condition for convergence is the coefficient matrix is diagonally dominant	Condition for convergence is the co-efficient matrix is diagonally dominant.

8. Solve 3x + 2y = 4, 2x – 3y = 7 by Gauss elimination method.

Solution:

Given 3x + 2y = 4

2x – 3y = 7

The given system is equivalent to

$$\begin{bmatrix} 3 & 2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

Here $[A,B] = \left(\begin{matrix} 3 & 2 \\ 2 & -3 \end{matrix}\middle|\begin{matrix} 4 \\ 7 \end{matrix}\right)$

$$= \left(\begin{matrix} 3 & 2 \\ 0 & -13 \end{matrix}\middle|\begin{matrix} 4 \\ 13 \end{matrix}\right) R_2 \leftrightarrow 3R_2 - 2R_1$$

This is an upper triangular matrix

Using backward substitution method

$-13y = 13$

$y = -1$

$3x + 2y = 4$

$3x - 2 = 4$

$3x = 6$

$x = 2$

Hence the solution is x = 2 and y = -1

9. Which iterative method converges faster for solving linear system of equations? Why?

Sol:

Gauss Seidal method is solving for linear system of equations converge faster. In this method the rate of convergence is roughly twice as fast as that of Gauss- Jacobi's method.

10.Write the uses of power method?

Sol:

To find the numerically largest eigen value of the given matrix.

11.

EIGEN VALUE OF A MATRIX BY POWER METHOD

12. Find the dominant eigen value and eigenvector of the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ by power method.

Solution:

$$Let\ X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$AX_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} = 7 \begin{pmatrix} 0.43 \\ 1 \end{pmatrix} = 7X_2$$

$$AX_2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0.43 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.43 \\ 5.29 \end{pmatrix} = 5.29 \begin{pmatrix} 0.46 \\ 1 \end{pmatrix} = 5.29X_3$$

$$AX_3 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0.46 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.46 \\ 5.38 \end{pmatrix} = 5.38 \begin{pmatrix} 0.46 \\ 1 \end{pmatrix} = 5.38X_4$$

Hence the dominant eigen value=5.38

The corresponding eigen vector= $\begin{pmatrix} 0.46 \\ 1 \end{pmatrix}$.

PART – B

FIXED POINT ITERATION METHOD

1. Using fixed point iteration method to find the positive root of the equation $\cos x - 3x + 1 = 0$.

Sol:

2. Solve $e^x - 3x = 0$ by method of fixed point iteration.

Sol:

NEWTON'S METHOD (OR) NEWTON RAPHSON METHOD

3. **Solve the equation $xlog_{10} x = 1.2$ using Newton-Raphson method.**

Solution:

Let $f(x) = xlog_{10}x - 1.2 \Rightarrow f'(x) = x \times \frac{1}{x} log_{10}e + log_{10}x$

$$f'(x) = log_{10}e + log_{10}x$$

$f(0) = 0\ log_{10}(0) - 1.2 = -1.2 = -ve$

$f(1) = 1\ log_{10}(1) - 1.2 = -1.2 = -ve$

$f(2) = 2\ log_{10}(2) - 1.2 = -0.598 = -ve$

$f(3) = 3\ log_{10}(3) - 1.2 = 0.231 = +ve$

Therefore the root lies between 2 & 3

$$|f(2)| > |f(3)|$$

Hence the root is nearer to 3choose $x_0 = 2.7$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.7 - \frac{f(2.7)}{f'(2.7)} = 1 - \left[\frac{2.7\ log_{10}(2.7) - 1.2}{log_{10}e + log_{10}2.7}\right] = 2.7 - \left[\frac{-0.035}{0.867}\right]$$

$x_1 = 2.740$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.740 - \frac{f(2.740)}{f'(2.740)} = 2.740 - \left[\frac{-0.006}{0.872}\right]$$

$x_2 = 2.741$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.741 - \frac{f(2.741)}{f'(2.741)} = 2.741 - \left[\frac{-.003}{0872}\right]$$

$x_3 = 2.741$

We observe that the root $x_2 = x_3 = 2.741$Correct to 3 decimal places. Hence the required root correct to three decimal places is 2.741

4. **Find the real positive root of $3x - cos\ x - 1 = 0$ by Newton's method correct to 5 decimal places.**

Solution :

Let $f(x) = 3x - cos\, x - 1$

$$f'(x) = 3 + sin\, x$$

$$f(0) = 0 - 1 - 1 = -2 = -ve$$

$$f(1) = 3 - cos\, 1 - 1 = 2 - cos\, 1 = 1.459698 = +ve$$

Therefore a root lies between 0 and 1.

$$|f(0)| > |f(1)|$$

Hence the root lies between 0 and 1.

$$Let\ \ x_0 = 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \dots \dots \dots (1)$$

Let n=0 in equation (1)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{f(1)}{f'(1)}$$

$$= 1 - \left[\frac{3(1) - \cos(1) - 1}{3 + \sin(1)}\right]$$

$$= 1 - \left[\frac{1.45970}{3.84147}\right]$$

$$= 1 - 0.37998$$

$$x_1 = 0.62002$$

Let n=1 in equation (1)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.62002 - \frac{f(0.62002)}{f'(0.62002)}$$

$$= 0.62002 - \left[\frac{3(0.62002) - \cos(0.62002) - 1}{3 + \sin(0.62002)}\right]$$

$$x_2 = 0.60712$$

Let n=2 in equation (1)

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.60712 - \frac{f(0.60712)}{f'(0.607102)}$$

$$= 0.60712 - \left[\frac{3(0.60712) - \cos(0.60712) - 1}{3 + \sin(0.60712)}\right]$$

$$x_3 = 0.60710$$

Let n=3 in equation (1)

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 0.60710 - \frac{f(0.60710)}{f'(0.60710)}$$

$$= 0.60710 - \left[\frac{3(0.60710) - \cos(0.60710) - 1}{3 + \sin(0.60710)}\right]$$

$$x_3 = 0.60710$$

From x_2 and x_3 we findout the root is 0.60710 correct to five decimal places.

5. Interpret the Newton's iterative formula to calculate the reciprocal of N and hence find the value of 1/26.

Sol:

Let $x = \frac{1}{N}$

$$N = \frac{1}{x}$$

$$\frac{1}{x} - N = 0$$

$f(x) = \frac{1}{x} - N$; $f'(x) = -\frac{1}{x^2}$

The Newton's formula is $\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$

$$= x_n - \frac{\left(\frac{1}{x_n} - N\right)}{\left(-\frac{1}{x_n^2}\right)}$$

$$= x_n + \left(\frac{1}{x_n} - N\right) X \quad x_n^2$$

$$= x_n + x_n - x_n^2 N$$

$$= x_n(2 - Nx_n)$$

SOLUTION OF LINEAR SYSTEM BY GAUSSIAN ELIMINATION METHOD

6. Solve the system of equations using Gauss elimination method $x+2y-5z=-9; 3x-y+2z=5; 2x+3y-z=3.$

Solution:

The given system is equivalent to

7. Solve the following equations by Gauss elimination method:

$3x - y + 2z = 12;\ x + 2y + 3z = 11; 2x - 2y - z = 2.$

Solution:

The given system is equivalent to

SOLUTION OF LINEAR SYSTEM BY GAUSS – JORDAN METHODS

8. **Using the Gauss – Jordan method solve the following equations 10x + y + z = 12, 2x + 10y + z = 13, x + y +5z =7**

Solution: **(AU N/D 2010)**

Given $10x + y + z = 12$

$2x + 10y + z = 13$

$x + y + 5z = 7$

Interchanging the first and the last equation then

$x + y + 5z = 7$

$2x + 10y + z = 13$

$10x + y + z = 12$

The given system is equivalent to

$$\begin{bmatrix} 1 & 1 & 5 \\ 2 & 10 & 1 \\ 10 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 13 \\ 12 \end{bmatrix}$$

AX = B

$$\text{Here } [A,\ B] = \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{array}\right)$$

Fix the pivot element row and make the other elements zero in the pivot element column.

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{array}\right) R_2 \leftrightarrow R_2 - 2R_1 \ \& R_3 \leftrightarrow R_3 - 10R_1$$

$$\sim \left(\begin{array}{ccc|c} 8 & 0 & 49 & 57 \\ 0 & 8 & -9 & -1 \\ 0 & 0 & 473 & 473 \end{array}\right) R_1 \leftrightarrow 8R_1 - R_2 \ \& R_3 \leftrightarrow 8R_3 + 9R_2$$

$$\sim\left(\begin{array}{ccc|c}8 & 0 & 49 & 57\\ 0 & 8 & -9 & -1\\ 0 & 0 & 1 & 1\end{array}\right) R_3 \leftrightarrow \frac{R_3}{473}$$

$$\sim\left(\begin{array}{ccc|c}8 & 0 & 0 & 8\\ 0 & 8 & 0 & 8\\ 0 & 0 & 1 & 1\end{array}\right) R_1 \leftrightarrow R_1 - 49R_3 \ \& R_2 \leftrightarrow R_2 + 9R_3$$

$$\sim\left(\begin{array}{ccc|c}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1\end{array}\right) R_1 \leftrightarrow \frac{R_1}{8} \ \& R_2 \leftrightarrow \frac{R_2}{8}$$

Therefore the solution is $\boldsymbol{x = 1, y = 1, z = 1}$

9. Using the Gauss – Jordan method solve the following equations $\boldsymbol{2x - y + 3z = 8}$, $\boldsymbol{-x + 2y + z = 4, 3x + y - 4z = 0}$ (AU M/J 2009)

Solution:

Given $2x - y + 3z = 8$

$$-x + 2y + z = 4$$

$$3x + y - 4z = 0$$

The given system is equivalent to

$$\begin{bmatrix}2 & -1 & 3\\ -1 & 2 & 1\\ 3 & 1 & -4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}8\\ 4\\ 0\end{bmatrix}$$

$AX = B$

$$\text{Here } [A,\ B] = \left(\begin{array}{ccc|c}2 & -1 & 3 & 8\\ -1 & 2 & 1 & 4\\ 3 & 1 & -4 & 0\end{array}\right)$$

Fix the pivot element row and make the other elements zero in the pivot element column.

$$\sim\left(\begin{array}{ccc|c}2 & -1 & 3 & 8\\ 0 & 3 & 5 & 16\\ 0 & 5 & -17 & -24\end{array}\right) R_2 \leftrightarrow 2R_2 + R_1 \ \& R_3 \leftrightarrow 2R_3 - 3R_1$$

$$\sim\left(\begin{array}{ccc|c}6 & 0 & 14 & 40\\ 0 & 3 & 5 & 16\\ 0 & 0 & -76 & -152\end{array}\right) \begin{array}{l}R_1 \leftrightarrow 3R_1 + R_2\\ R_3 \leftrightarrow 3R_3 - 5R_2\end{array}$$

$$\sim\left(\begin{array}{ccc|c}6 & 0 & 14 & 40\\ 0 & 3 & 5 & 16\\ 0 & 0 & 1 & 2\end{array}\right) R_3 \leftrightarrow \frac{R_3}{-76}$$

$$\sim\left(\begin{array}{ccc|c}6 & 0 & 0 & 12\\ 0 & 3 & 0 & 6\\ 0 & 0 & 1 & 2\end{array}\right) R_1 \leftrightarrow R_1 - 14R_3 \ \& R_2 \leftrightarrow R_2 - 5R_3$$

$$\sim\left(\begin{array}{ccc|c}1 & 0 & 0 & 2\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 2\end{array}\right) \quad R_1 \leftrightarrow \frac{R_1}{6} \;\&\; R_2 \leftrightarrow \frac{R_2}{3}$$

Therefore the solution is $\boldsymbol{x = 2, y = 2, z = 2}$

GAUSS – JACOBI METHOD AND GAUSS – SEDIAL METHOD

10. Solve the system of equation by Gauss – Sedial method correct to 4 decimal places

20x + y –2 z = 17, 3x + 20y -z = -18, 2x -3y +20z = 25

Solution: .

Given 20x + y –2 z = 17
3x + 20y -z = -18
2x -3y +20z = 25

As the coefficient matrix is diagonally dominant solving for x, y, z we get

$$x = \frac{1}{20}[17 - y + 2z], y = \frac{1}{20}[-18 - 3x + z], z = \frac{1}{20}[25 - 2x + 3y]$$

Let the initial value be y=0, z=0

Iteration	$x = \left[\frac{17 - y + 2z}{20}\right]$	$y = \left[\frac{-18 - 3x + z}{20}\right]$	$z = \left[\frac{25 - 2x + 3y}{20}\right]$
1	0.85	-1.0275	1.0109
2	1.0025	-0.9998	0.9998
3	1.0000	-1.0000	1.0000
4	1	-1	1

Hence x = 1, y = -1, z = 1.

11. Solve the system of equation by Gauss – Seidel method 28x +4 y – z = 32, x + 3y +10z = 24, 2x +17y +4z = 35.

Solution: .

Given $28x + 4y - z = 32$

$x + 3y + 10z = 24$

$2x + 17y + 4z = 35$

As the coefficient matrix is diagonally dominant solving for x, y, z we get

$$x = \frac{1}{28}[32 - 4y + z]$$

$y = \frac{1}{17}[35 - 2x - 4z]$

$z = \frac{1}{20}[24 - x - 3y]$

Let the initial value be y=0, z=0

Let the initial value be y=0, z=0

Iteration	$x = \left[\frac{32 - 4y + z}{28}\right]$	$y = \left[\frac{35 - 2x - 4z}{17}\right]$	$z = \left[\frac{24 - x - 3y}{20}\right]$
1	1.1429	1.9244	1.8084
2	0.9325	1.5236	1.8497
3	0.9913	1.5070	1.8488
4	0.9936	1.5069	1.8486
5	0.9936	1.5069	1.8486

Hence x =0.9936 , y = 1.5069, z = 1.8486

EIGEN VALUE OF A MATRIX BY POWER METHOD

12. Find the largest Eigen value and the corresponding Eigen vector of $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ using power method. Using $x_1 = (1 \quad 0 \quad 0)^T$ as initial vector.

Solution:

Let $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an approximate eigen value.

$$AX_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\, X_2$$

$$AX_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = 7\, X_3$$

$$AX_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.5714 \\ 1.8572 \\ 0 \end{bmatrix} = 3.5714\begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = 3.5714\, X_4$$

$$AX_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = 4.12 \begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = 4.12\, X_5$$

$$AX_5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9706 \\ 1.9902 \\ 0 \end{bmatrix} = 3.9706 \begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = 3.9706\, X_6$$

$$AX_6 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.0072 \\ 2.0024 \\ 0 \end{bmatrix} = 4.0072 \begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = 4.0072\, X_6$$

$$AX_7 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9982 \\ 1.9994 \\ 0 \end{bmatrix} = 3.9982 \begin{bmatrix} 1 \\ 0.5000 \\ 0 \end{bmatrix} = 3.9982\, X_8$$

$$AX_8 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = 4\, X_9$$

$$AX_9 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

Therefore Dominant eigen value =4; corresponding eigen vector is (1, 0.5, 0)

13. Find , by power method, the largest Eigen value and the corresponding Eigen vector of a matrix $A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$ with initial vector $(\mathbf{1}\ \ \mathbf{1}\ \ \mathbf{1})^{T}$.

Solution:

Let $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be an arbitrary initial eigen vector.

$$AX_1 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 13 \end{bmatrix} = 13 \begin{bmatrix} 0.231 \\ 0.692 \\ 1 \end{bmatrix} = 13X_2$$

$$AX_2 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.231 \\ 0.692 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.307 \\ 6.077 \\ 12.537 \end{bmatrix} = 12.537 \begin{bmatrix} 0.104 \\ 0.485 \\ 1 \end{bmatrix} = 12.537X_3$$

$$AX_3 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.104 \\ 0.485 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.559 \\ 5.282 \\ 11.836 \end{bmatrix} = 11.836 \begin{bmatrix} 0.047 \\ 0.485 \\ 1 \end{bmatrix} = 11.836X_4$$

$$AX_4 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.047 \\ 0.485 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.385 \\ 5.033 \\ 11.737 \end{bmatrix} = 11.737 \begin{bmatrix} 0.033 \\ 0.429 \\ 1 \end{bmatrix} = 11.737X_5$$

$$AX_5 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.033 \\ 0.429 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.32 \\ 4.957 \\ 1 \end{bmatrix} = 11.683 \begin{bmatrix} 0.027 \\ 0.424 \\ 1 \end{bmatrix} = 11.683X_6$$

$$AX_6 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.027 \\ 0.424 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.299 \\ 4.929 \\ 11.669 \end{bmatrix} = 11.669 \begin{bmatrix} 0.026 \\ 0.422 \\ 1 \end{bmatrix} = 11.669X_7$$

$$AX_7 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.026 \\ 0.422 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.299 \\ 4.922 \\ 11.662 \end{bmatrix} = 11.662 \begin{bmatrix} 0.025 \\ 0.422 \\ 1 \end{bmatrix} = 11.662 X_8$$

$$AX_8 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.025 \\ 0.422 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.291 \\ 4.919 \\ 11.663 \end{bmatrix} = 11.663 \begin{bmatrix} 0.025 \\ 0.422 \\ 1 \end{bmatrix}$$

Therefore, the dominant eigenvector is $\begin{bmatrix} 0.025 \\ 0.422 \\ 1 \end{bmatrix}$, eigenvalue is 11.663.

14. Find the dominant eigen value and its eigenvector of the matrix by power method

$$\boldsymbol{A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}}.$$

(OR)

Using power method, find all the eigen values of $\boldsymbol{A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}}$.

Solution:

Let $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an initial eigen vector.

$$AX_1 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{5} \\ 0 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0 \\ 0.02 \end{bmatrix} = 5\, X_2$$

$$AX_2 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.02 \end{bmatrix} = \begin{bmatrix} 5.2 \\ 0 \\ 2 \end{bmatrix} = 5.2 \begin{bmatrix} 1 \\ 0 \\ 0.3846 \end{bmatrix} = 5.2\, X_3$$

$$AX_3 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.3846 \end{bmatrix} = \begin{bmatrix} \mathbf{5.3846} \\ 0 \\ 2.9231 \end{bmatrix} = 5.3846 \begin{bmatrix} 1 \\ 0 \\ 0.5429 \end{bmatrix} = 5.3846\, X_4$$

$$AX_4 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.5429 \end{bmatrix} = \begin{bmatrix} \mathbf{5.5429} \\ 0 \\ 3.7143 \end{bmatrix} = 5.5429 \begin{bmatrix} 1 \\ 0 \\ 0.6701 \end{bmatrix} = 5.5429\, X_5$$

$$AX_5 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.6701 \end{bmatrix} = \begin{bmatrix} \mathbf{5.6701} \\ 0 \\ 0.4305 \end{bmatrix} = 5.6701 \begin{bmatrix} 1 \\ 0 \\ 0.7672 \end{bmatrix} = 5.6701\, X_6$$

Continuing in the same way, we can observe that 15^{th} and 16^{th} iterations are equal. In that case

$$AX_{16} = \begin{bmatrix} 5.997 \\ 0 \\ 5.985 \end{bmatrix} = 5.997 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore the eigen value $\lambda = 6$ and eigen vector $X = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$B = A - 6I$$

$$B = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Now take the initial vector of B as $Y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$BY_1 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$BY_2 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$BY_3 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

The dominant eigen value of $B = -2$ and eigen vector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

$\therefore$ the smallest eigen value of $a = -2 + 6 = 4$

By using the property,

Sum of the Eigen values=Trace of A

$$= 5 - 2 + 5 = 8$$

The third eigen value is $\lambda_1 + \lambda_2 + \lambda_3 = 8$

$6 + 4 + \lambda_3 = 8$

$\lambda_3 = -2$

Therefore the eigen values are 6,4,-2

UNIT-II

INTERPOLATION AND APPROXIMATION

PART-A

LAGRANGE'S INTERPOLATION

1. **Write down the Lagrange's Interpolation formula.**

Solution:

Let y = f(x) be a function which takes the values $y_0, y_1, y_2, \ldots\ldots y_n$ corresponding to $x_0, x_1, x_2, \ldots. x_n$

Then Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} y_1$$

$$+ \ \ldots\ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} y_n$$

2. **Find the second degree polynomial through the points (0,2),(2,1),(1,0) using Lagrange's formula.**

Solution:

We use Lagrange's interpolation formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$= \frac{(x-2)(x-1)}{(0-2)(0-1)}.2 + \frac{(x-0)(x-1)}{(2-0)(2-1)}.1 + \frac{(x-0)(x-2)}{(1-0)(1-2)}.0$$

$$= x^2 - 3x + 2 + \frac{1}{2}(x^2 - x) = \frac{1}{2}(2x^2 - 6x + 4 + x^2 - x)$$

$$y = \frac{1}{2}(3x^2 - 7x + 4)$$

DIVIDED DIFFERENCES

3. Distinguish between interpolation and extrapolation.

Solution:

Interpolation	**Extrapolation**
To find the values of a function inside a given range is interpolation.	To find the values of a function outside a given range is extrapolation.

4. Find the divided difference of f(x) which takes the values 1, 4, 40, 85 with arguments 0, 1, 3, 4

Solution:

The divided difference table is as follows

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		$\frac{4-1}{1-0} = 3$		
			$\frac{18-3}{3-0} = 5$	
1	4			
		$\frac{40-4}{3-1} = 18$		$\frac{6.75-5}{4-0} = 0.44$
			$\frac{45-18}{4-0} = 6.75$	
3	40			
		$\frac{85-40}{4-3} = 45$		
4	85			

5. Find the divided differences of $f(x) = x^3 + x + 2$ for the arguments 1, 3 , 6, 11.

Solution: **(AU A/M 2011)**

$$f(1) = 1^3 + 1 + 2 = 4$$

$$f(3) = 3^3 + 3 + 2 = 32$$

$$f(6) = 6^3 + 6 + 2 = 224$$

$$f(11) = 11^3 + 11 + 2 = 1344$$

The divided difference table is as follows

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	4			
		$\frac{32-4}{3-1} = 14$		
			$\frac{64-14}{6-1} = 10$	
3	32			
		$\frac{224-32}{6-3} = 64$		$\frac{20-10}{11-1} = 1$
			$\frac{224-64}{11-3} = 20$	
6	224			
		$\frac{1344-224}{11-6} = 224$		
11	1344			

CUBIC SPLINE

6. Define cubic spline function.

Solution:

We define a cubic spline, $S(x)$ as follows:

i) $S(x)$ is a polynomial of degree one for $x < x_0$ and $x > x_n$.

ii) $S(x)$ is at most a cubic polynomial in each interval $(x_i - 1, x_{i+1})$, $i = 1,2, \dots n$.

iii) $S(x)$, $S'(x)$ and $S''(x)$ are continuos at each point (xi, yi), $i = 0,1,2 \dots n$ and

iv) $S(x_i) = y_i$, $i = 0,1,2, \dots n$.

NEWTON'S FORWARD AND BACKWARDINTERPOLATION

7. Derive Newton's backward interpolation formula using operator method.

(OR) State Newton's backward formula for interpolation.

State Newton's backward difference formula.

Solution:

$$y(x) = y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \cdots$$

Where $v = \frac{x - x_n}{h}$

8. Derive Newton's forward interpolation formula using equal intervals .

Solution:

$$y_n = f(x_0 + n\text{h}) = y_0 + n\Delta y_n + \frac{n(n-1)}{2!}\Delta^2 y_n + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_n + + \cdots$$

9. Find the first and second divided difference with arguments $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ of the function $\boldsymbol{f}(\boldsymbol{x}) = \frac{\mathbf{1}}{\boldsymbol{x}}$

Solution:

If $f(x) = \frac{1}{x} \Rightarrow f(a) = \frac{1}{a}$

$$f(a,b) = \Delta\left[\frac{1}{a}\right] = \frac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\frac{1}{ab} \qquad \left[\because f(x_0,x_1) = \frac{f(x_1)-f(x_0)}{x_0-x_1}\right]$$

$$f(a,b,c) = \frac{f(b,c)-f(a,b)}{c-a} = \frac{-\frac{1}{bc}-\frac{1}{ab}}{c-a} = \frac{1}{abc}\left[\frac{c-a}{c-a}\right] = \frac{1}{abc}$$

$\therefore \qquad \Delta^2\left[\frac{1}{a}\right] = \frac{1}{abc}$

10. When to use Newton's forward interpolation and when to use Newton's backward interpolation?

Solution:

This formula is used to interpolate the values of y near the beginning of the table value and also for extrapolating the values of y short distance ahead (to the left) of y_0.

(i) Thus formula is used mainly to interpolate the values of y near to end of the set of tabular values and also for extrapolating the values of y short distance ahead (to the right) of y_0.

PART-B

LAGRANGE'S INTERPOLATION

1. Find the interpolation polynomial f(x) by Lagrange's formula and hence find f(3) for (0,2),(1,3),(2,12)and (5,147). (OR)

Find the polynomial $f(x)$ by using Lagrange's formula and hence find $f(3)$ for

x	0	1	2	5
$f(x)$	2	3	12	147

Solution:

By Lagrange's interpolation formula, we have

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y = f(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$

$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$$= \frac{(x-1)(x-2)(x-5)}{(-10)}(2) + \frac{x(x-2)(x-5)}{4}(3)$$

$$+ \frac{(x-1)(x-5)}{-6}(12) + \frac{x(x-1)(x-2)}{60}(147)$$

Put x = 3 we get

$$y = f(3) = \frac{(3-1)(3-2)(3-5)}{-10}(2) + \frac{3(3-2)(3-5)}{4}(3)$$

$$+ \frac{3(3-1)(3-5)}{-6}(12) + \frac{3(3-1)(3-2)}{60}(147)$$

$$= \frac{2(-2)}{(-10)}(2) + \frac{3(-2)}{4}(3) + \frac{3(2)(-2)}{(-6)}(12) + \frac{3(2)}{60}(147)$$

$$y = f(3) = \frac{4}{10}(2) - \frac{6}{4}(3) + 2(12) + \frac{1}{10}(147) = \frac{8}{10} - \frac{18}{4} + 24 + \frac{147}{10}$$

$$f(x) = 35$$

2. Use Lagrange's formula to construct a polynomial which takes the values

$f(0) = -12, f(1) = 0, f(3) = 6$ **and** $f(4) = 12.$ **Hence find** $f(2).$

Solution:

By Lagrange's interpolation formula, we have

$$y = f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3$$

$$y = f(x) = \frac{(x - 1)(x - 3)(x - 4)}{(0 - 1)(0 - 3)(0 - 4)}(-12) + 0$$

$$+ \frac{(x - 0)(x - 1)(x - 4)}{(3 - 0)(3 - 1)(3 - 4)}(6) + \frac{(x - 0)(x - 1)(x - 3)}{(4 - 0)(4 - 1)(4 - 3)}(12)$$

$$= \frac{(x - 1)(x - 3)(x - 4)}{(-12)}(-12) + \frac{x(x - 1)(x - 4)}{(-6)}(6)$$

$$+ \frac{x(x - 1)(x - 3)}{12}(12)$$

$$= (x - 1)(x - 3)(x - 4) - x(x - 1)(x - 4) + x(x - 1)(x - 3)$$

$$= (x - 1)[x^2 - 3x - 4x + 12 - x^2 + 4x + x^2 - 3x]$$

$$= (x - 1)(x^2 - 6x + 12)$$

$$= x^3 - 6x^2 + 12x - x^2 + 6x - 12$$

$$y(x) = x^3 - 7x^2 + 18x - 12$$

$$\therefore y(2) = 2^3 - 7(2)^2 + 18(2) - 12 = 4$$

$$\therefore y(2) = 4$$

DIVIDED DIFFERENCES

3. Determine $f(x)$ as a polynomial in x for the following data, using Newton's divided difference formula. Also find $f(2)$

x	-4	-1	0	2	5
$f(x)$	1245	33	5	9	1335

Solution:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-4	1245				
		$\frac{33-124}{-1-(-4)} = -404$			
-1	33		$\frac{-28-(-404)}{0-(-4)} = 94$		
		$\frac{5-33}{0-(-1)} = -28$		$\frac{10-94}{2-(-4)} = -14$	
0	5		$\frac{2-(-28)}{2-(-1)} = 10$		
		$\frac{9-5}{2-0} = 2$			$\frac{13+14}{5-(-4)} = 3$
				$\frac{88-10}{5-(-1)} = 13$	
			$\frac{442-2}{5-0} = 88$		
2	9				
		$\frac{1335-9}{5-2} = 442$			
5	1335				

By Newton's divided difference interpolation formula,

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3)$$

$$\Rsh \cdots \ldots \ldots \ldots (1)$$

Here $f(x_0) = 1245, f(x_0, x_1) = -404, f(x_0, x_1, x_2) = 94, f(x_0, x_1, x_2, x_3) = -14$ & $f(x_0, x_1, x_2, x_3, x_4)$ =3,

Hence we using this formula in equation (1) we get

$$f(x) = 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94) + (x + 4)(x + 1)(x - 0)(-14)$$

$$+ (x + 4)(x + 1)(x - 0)(x - 2)(3)$$

$$= 1245 - 404x - 1616 + 94x^2 + 470x + 376 - 14x^3 - 70x^2 - 56x$$

$$+ 3x[x^3 - 2x^2 + 5x^2 - 10x + 4x - 8]$$

$$= -14x^3 + 24x^2 + 10x + 5 + 3x[x^3 \mp 3x^2 - 6x - 8]$$

$$= -14x^3 + 24x^2 + 10x + 5 + 3x^4 + 9x^3 - 18x^2 - 24x$$

$$\therefore \quad f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5$$

$$\Rightarrow \quad f(2) = 3 \times 2^4 - 5 \times 2^3 + 6 \times 2^2 - 14 \times 2 + 5 = 48 - 40 + 24 - 28 + 5$$

$$\therefore \quad f(2) = 9$$

4. Use Newton's divided difference formula find f(9) given the values (5,150), (7,392), (13,2366) and (17,5202)

Solution:

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$

5	150			
		$\frac{392-150}{7-5}=121$		
			$\frac{329-121}{13-5}=26$	
7	392			
		$\frac{2366-392}{13-7}=329$		$\frac{38-26}{17-5}=1$
			$\frac{709-329}{17-7}=38$	
13	2366			
		$\frac{5202-2366}{17-13}=709$		
17	5202			

By Newton's divided difference formula

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$

$$+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)\ \dots\dots(1)$$

$$=150+(x-5)(121)+(x-5)(x-7)(26)+(x-5)(x-7)(x-13)(1)$$

$$f(9)=150+(9-5)(121)+(9-5)(9-7)(26)+(9-5)(9-7)(9-13)(1)$$

$$f(9)=150+484+192-32$$

$f(9)=794$

CUBIC SPLINE

5. The following values of x and y are given in table:

$\boldsymbol{x}$:	**1**	**2**	**3**	**4**
$\boldsymbol{y}$:	**1**	**2**	**5**	**11**

Find the cubic splines and evaluate$y(1.5)$.

Solution:

Here $h = 1, n = 3$, also assume $M(0) = M(3) = 0$.

$h \rightarrow length\ of\ the\ interval$

$n \rightarrow number\ of\ intervals$

We know that the cubic spline polynomial is

$$Y = s(x) = \frac{1}{6h}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i] + \frac{1}{h}(x_i - x)\left[y_{i-1} - \frac{h^2}{6}M_{i-1}\right]$$

$$+\frac{1}{h}(x - x_{i-1})\left[y_i - \frac{h^2}{6}M_i\right] ----(1)$$

Here $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$

$y_0 = 1$, $y_1 = 2$, $y_2 = 5$, $y_3 = 11$

$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}]$ $for\ i = 1,2, \ldots (n-1)$ $\ldots.(2)$

$i.e;\ i = 1,2$ $[\therefore n = 3]$

$M_0 = y''_0 = 0,\ M_3 = y''_3 = 0$

$$(1) \rightarrow M_0 + 4M_1 + M_2 = 6[y_0 - 2y_1 + y_2] = 6(1 - 2(2) + 5) = 12$$

$$4M_1 + M_2 = 6[1 - 2(5) + 11] = 180$$

$$4M_1 + M_2 = 12 \quad [since M_0 = 0] \quad \ldots\ldots\ldots(3)$$

$$For\ i = 2 \Rightarrow M_1 + 4M_2 + M_3 = 6[y_1 - 2y_2 + y_3]$$

$$= 6[\,2 - 2(5) + 11] = 18$$

$$M_1 + 4M_2 = 18 \ldots\ldots\ldots.(4) \quad \because M_3 = 0]$$

$$(3) \times 4 \Rightarrow 16M_1 + 4M_2 = 48 \ldots\ldots..(5)$$

$$(5) - (4) \Rightarrow M_1 + 4M_2 = 18$$

$$15\,M_1 = 30$$

$$M_1 = 15$$

$$(3) \Rightarrow 4(2) + M_2 = 12$$

$$M_2 = 4$$

For $i = 1$, we get the cubic spline, for $0 \leq x \leq 1$, is given by

$$s(x) = \frac{1}{6}[(2-x)^3(0) + (x-1)^3(2)] + (2-x)\left(1 - \frac{1}{6}(0)\right) + (x-1)\left(2 - \frac{1}{6}(2)\right)$$

$$= \frac{1}{6}[(x-1)^3(2)] + (2-x)(1) + (x-1)\left(2 - \frac{1}{3}\right)$$

$$= \frac{1}{3}(x-1)^3 + (2-x) + (x-1)\left(\frac{5}{3}\right)$$

$$= \frac{1}{3}[x^3 - 3x^2 + 3x - 1] + (2-x) + (x-1)\left(\frac{5}{3}\right)$$

$$= \frac{1}{3}[x^3 - 3x^2 + 3x - 1 + 6 - 3x + 5x - 5]$$

$$s(x) = y(x) = \tfrac{1}{3}[x^3 - 3x^2 + 5x] ----(6)$$

For $i = 2$, we get the cubic spline, for $1 \le x \le 2$, is given by

$$s(x) = \tfrac{1}{6}[(3-x)^3(2) + (x-2)^3(4)] + (3-x)\left(2 - \tfrac{1}{6}(2)\right)$$

$$+(x-2)\left(5 - \frac{1}{6}(4)\right)$$

$$s(x) = y(x) = \tfrac{1}{3}[x^3 - 3x^2 + 5x] ----- (7)$$

For $i = 3$, we get the cubic spline, for $2 \le x \le 3$, is given by

$$s(x) = \tfrac{1}{6}[(4-x)^3(4) + (x-3)^3(0)] + (4-x)\left(5 - \tfrac{1}{6}(4)\right) + (x-3)\left(11 - \tfrac{1}{6}(0)\right)$$

$$s(x) = y(x) = \tfrac{1}{3}[-2x^3 + 24x^2 - 76x + 81] ----- (8)$$

Equation (6), (7) &(8) gives the cubic spline in each sub-interval.

To find $y(1.5)$

$$(7) \quad \Rightarrow \quad y(1.5) = \frac{1}{3}[(1.5)^3 - 3(1.5)^2 + 5(1.5)]$$

$$y(1.5) = \tfrac{1}{3}[4.125] \qquad since \quad 1.5\ lies\ in\ 1 \le x \le 2]$$

$y(1.5) = 1.375$

6. Find the Cubic Spline approximation for the function given below.

x	0	1	2	3
$f(x)$	1	2	33	244

Assume that $M(0) = 0 = M(3)$. Hence find the value of $f(2.5)$.

Solution:

Here $h = 1, n = 3$, also assume $M(0) = M(3) = 0$.

$h \to length\ of\ the\ interval$

$n \to number\ of\ intervals$

We know that the cubic spline polynomial is

$$Y = s(x) = \frac{1}{6h}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i] + \frac{1}{h}(x_i - x)\left[y_{i-1} - \frac{h^2}{6}M_{i-1}\right]$$

$$+\frac{1}{h}(x - x_{i-1})\left[y_i - \frac{h^2}{6}M_i\right]$$

$\hookrightarrow \cdots \dots \dots \dots \dots (1)$

We have

$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}] \quad for\ i = 1,2, \dots (n-1) \quad \dots .(2)$

$i.e;\ i = 1,2 \quad [\therefore n = 3]$

$M_0 = y''_0 = 0,\ M_3 = y''_3 = 0$

$(1) \to M_0 + 4M_1 + M_2 = 6[y_0 - 2y_1 + y_2] = 6(1 - 4 + 33) = 180$

$4M_1 + M_2 = 6[1 - 2(5) + 11] = 180$

$4M_1 + M_2 = 180 \qquad \dots \dots \dots (3)$

$For\ i = 2 \Rightarrow M_1 + 4M_2 + M_3 = 6[y_1 - 2y_2 + y_3]$

$= 6[2 - 66 + 244] = 1080$

$M_1 + 4M_2 = 1080 \dots \dots \dots .(4) \qquad \because M_3 = 0]$

$$(2) \times 4 \quad \Rightarrow \quad 16M_1 + 4M_2 = 720 \ldots\ldots..(5)$$

$$(5) - (4) \Rightarrow 15M_1 = -360(5)$$

$$M_1 = -24$$

$$(1) \qquad \Rightarrow \qquad 4(-24) + M_2 = 180$$

$$M_2 = 276$$

For $i = 1$, we get the cubic spline, for $0 \leq x \leq 1$, is given by

$$s(x) = \frac{1}{6}[(1-x)^3(0) + (x-0)^3(-24)] + (1-x)\left(1 - \frac{1}{6}(0)\right) + (x-0)\left(2 - \frac{(-24)}{6}\right)$$

$$s(x) = y(x) = -4x^3 + (1-x) + 6x$$

$$s(x) = -4x^3 + 5x + 1 \ldots\ldots\ldots.(6)$$

For$i = 2$, we get the cubic spline, for $1 \leq x \leq 2$, is given by

$$s(x) = \frac{1}{6}[(2-x)^3(-24) + (x-1)^3(276)] + (2-x)\left(\frac{2-(-24)}{6}\right)$$

$$+(x-1)\left(33 - \frac{(276)}{6}\right)$$

$$s(x) = y(x) = 50x^3 - 162x^2 + 167x - 53 \ldots\ldots\ldots.(7)$$

For$i = 3$, we get the cubic spline, for $2 \leq x \leq 3$, is given by

$$s(x) = \frac{1}{6}[(3-x)^3(276) + (3-x)^3(33-46)] + (x-2)(244)$$

$$s(x) = y(x) = -46x^3 + 414x^2 - 985x + 715 \ldots\ldots(8)$$

Equation (6), (7) &(8) gives the cubic spline in each sub-interval.

To find $y(2.5)$

$$(8) \quad \Rightarrow \quad y(2.5) = -46(2.5)^3 + 414(2.5)^2 - 985(2.5) + 715$$

$$y(2.5) = 121.25 \qquad since \quad 2.5 \; lies \; in \; 2 \leq x \leq 3]$$

NEWTON'S FORWARD AND BACKWARD INTERPOLATION

7. Find a polynomial of degree two for the data by Newton's forward difference formula.

X	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
y	**1**	**2**	**4**	**7**	**11**	**16**	**22**	**29**

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		1	
		2		0
2	4		1	
		3		0
3	7		1	
		4		0
4	11		1	
		5		0
5	16		1	
		6		0
6	22		1	

7	29	7		

Here $x_0 = 0,\ y_0 = 1,\ \ h = 1$

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots\ldots$$

Where $u = \frac{x-x_0}{h} = \frac{x-0}{1} = x \Rightarrow u = x$

$$y(x) = 1 + x(1) + \frac{x(x-1)}{2!}(1)$$

$$= 1 + x + \frac{x^2-x}{2} = \frac{2+2x+x^2-x}{2}$$

$y(x) = \frac{1}{2}[x^2 + x + 2]$ is the required polynomial.

8. **Using Newton's forward interpolation formula, find the cubic polynomial which takes the following values .**

X	**0**	**1**	**2**	**3**
Y	**1**	**2**	**1**	**10**

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$

x_0 0	y_0 1			
		$2-1=1(\Delta y_0)$		
x_1 1	y_1 2		$-1-1$	
			$=-2\ (\Delta^2 y_0)$	
		$1-2=-1(\Delta y_1)$		$10+2$
x_2 2	y_2 1			$=12\ (\Delta^3 y_0)$
			$9+1 \quad =10\ (\Delta^2 y_1)$	
		$10-1=9(\Delta y_2)$		
x_3 3	y_3 10			

We will use forward difference formula

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

Where $u = \frac{x-x_0}{h} = \frac{x-0}{1} = x \Rightarrow u = x$

$$\therefore \qquad y(x) = 1 + x(1) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(12)$$

$$= 1 + x - \frac{x^2-x}{2}(2) + \frac{x(x-1)(x-2)}{6}(12)$$

$$= 1 + x - x^2 + x + 2x(x-1)(x-2)$$

$$= 1 + x - x^2 + x + 2x(x^2-3x+2)$$

$$= 1 + x - x^2 + x + 2x^3 - 6x^2 + 4x$$

$$= 1 + 6x - 7x^2 + 2x^3$$

$$\therefore \qquad y(x) = 2x^3 - 7x^2 + 6x + 1$$

9. **From the given table compute the value of sin 38° (M/J 2016)**

x	**0**	**10**	**20**	**30**	**40**
sin x	**0**	**0.17365**	**0.34202**	**0.5**	**0.64279**

Solution:

We form the difference table:

x	$Y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
	(y_n)				
0	0	(Δy_0)			
		0.17365	$(\Delta^2 y_0)$		
10	0.17365		-0.00528	$(\Delta^3 y_0)$	
		0.16837		-0.00511	$(\Delta^4 y_0)$
20	0.34202		-0.01039		0.00031
		0.15798		-0.00487	$(\nabla^4 y_n)$
30	0.5		-0.01519	$(\nabla^3 y_n)$	
		0.14279	$(\nabla^2 y_n)$		
40	0.64279				
(x_n)	(y_n)				

We will use backward difference formula

$$y(x) = y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n \quad + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \cdots$$

Where $v = \frac{x - x_n}{h} = \frac{40 - 0.064279}{10} = -0.2$

$$y(38°) = 0.64279 \ \ - 0.028 - 0.0127 + 0.0290$$

$$y(38°) = 0.64249$$

$$\sin 38° = 0.61568$$

UNIT-III

NUMERICAL DIFFERENTIATION AND INTEGRATION

PART-A

DIFFERENTION USING INTERPOLATION FORMULA

1. **Write down the expression for $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *at* $x = x_n$ by Newton's backward difference formula**

Solution:

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n - \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

NUMERICAL INTEGRATION BY TRAPEZOIDAL METHOD

2. **State Trapezoidal rule to evaluate $\int_a^b f(x)dx$.**

Solution:

$$\int_a^b f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\right]$$

3. **Taking $h = 0.5$, evaluate $\int_1^2 \frac{dx}{1+x^2}$ using Trapezoidal rule.**

Solution:

Here $y(x) = \frac{1}{1+x^2}$

Length of the interval = 1

x	:	1	1.5	2
$y = \frac{1}{1+x^2}$	:	0.5	0.3077	0.2

$h = 0.5$

By Trapezoidal rule

Trapezoidal rule

$$= \frac{h}{2}[sum\ of\ the\ first\ and\ last\ ordinates]$$

$$+ 2[sum\ of\ the\ remaining\ ordinates]$$

$$\int_1^2 \frac{dx}{1+x^2} = \frac{h}{2}[(0.5 + 0.2) + 2(0.3077)]$$

$$\int_1^2 \frac{dx}{1+x^2} = \frac{0.5}{2}[0.7 + 0.6154]$$

$$\int_1^2 \frac{dx}{1+x^2} = \frac{0.5}{2}[1.3154] = 0.3289$$

4. Using Trapezoidal rule, evaluate $\int_0^{\pi} \sin x\, dx$ by dividing the range into 6 equal parts.

Solution:

5. **Evaluate** $\int_{\frac{1}{2}}^{1} \frac{1}{x}\,dx$ **by Trapezoidal rule, dividing the range into 4equal parts.**

Solution:

6. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule.

Solution:

Here $y(x) = \frac{1}{1+x^2}$

Length of the interval = 1

x	:	0	0.2	0.4	0.6	0.8	1
$y = \frac{1}{1+x^2}$	:	1	0.96154	0.86207	0.73529	0.60976	0.5

$h = 0.2$

By Trapezoidal rule

Trapezoidal rule

$$= \frac{h}{2}[sum\ of\ the\ first\ and\ last\ ordinates]$$

$$+ 2[sum\ of\ the\ remaining\ ordinates]$$

$$\int_{-1}^{1} \frac{dx}{1+x^2} = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{0.2}{2}[(1 + 0.5) + 2(0.96154 + 0.86207 + 0.9412 + 0.73529 + 0.60976)]$$

$$\int_{-1}^{1} \frac{dx}{1+x^2} = \frac{0.2}{2}[7.83732] = 0.783732 \ \ldots\ldots..(1)$$

By actual integration,

$$\int_0^6 \frac{dx}{1+x^2} = (tan^{-1}x)_0^1 = tan^{-1}1 - tan^{-1}0 = \frac{\pi}{4} \quad \ldots\ldots(2)$$

$From$ (1)& (2) $$\frac{\pi}{4} = 0.783732$$

$$\pi = 3.13493(approximately$$

NUMERICAL INTEGRATION BY SIMPSON'S 1/3 AND 3/8 RULES

7. **State Simpson's one-third rule.**

Solution:

Simpson's one third rule is

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}[(y_0+y_n) + 4(y_1+y_3+\cdots+y_{n-1}) \quad + 2(y_2+y_4+\cdots+y_{n-2})]$$

8. **State the local error term in Simpson's 1/3 rule.**

Solution:

The local error in the interval (x_0, x_2) is $\frac{-h^5}{90} y_0^{(4)} = \frac{-h^4}{180}(b-a)y_0^{(4)}$,

where $y_0^{(4)}$ is the 4th derivative of y=f(x) at $x = x_0$

9. **State Simpson's 3/8 rule of integration.**

Solution:

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-1})+2(y_3+y_6+\cdots+y_{n-3})]$$

ROMBERG'S METHOD

10. State Trapezoidal rule for evaluating $\int_a^b\int_c^d f(x,y)\ dx\ dy$.

Solution:

$I = \frac{hk}{4}[(Sum$ of values of f at Four corners) + 2(Sum of the values of f at remaining nodes on the boundary) $+\ 4$(sum of values of f at interior nodes)

11. State Romberg's integration formula to find the value of $I = \int_a^b f(x)\,dx$ for first two intervals.

Solution:

Let I_1 and I_2 be the values of the integral I, by trapezoidal rule with h,h/2 as width of interval .

Then Romberg's formula $I = I_2 + \left(\frac{I_2 - I_1}{3}\right)$

PART – B

DIFFERENTION USING INTERPOLATION FORMULA

1. Construct $\frac{dx}{dy}$ *and* $\frac{d^2y}{d^2x}$ at $x = 51$, from the following data:

X:	50	60	70	80	90
Y:	19.96	36.65	58.81	77.21	94.61

Solution:

Given x= 51, x_0 = 50 h = 60 – 50 = 10

$$u = \frac{x - x_0}{h} = \frac{51 - 50}{10} = 0.1$$

At x = 51, u = 0.1

Difference table

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
50	19.96				
		16.69			
60	36.65		5.47		
		22.16		-9.23	
70	58.81		-3.76		11.99
		18.40		2.76	
80	77.21		-1.00		
		17.40			
90	94.61				

W.K.T the Newton's forward difference formula is

$$f'(x) = \left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{dy}{dx}\right)_{u=0.1} = \frac{1}{h}[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2 - 6u + 2)}{3!}\Delta^3 y_0$$

$$+ \frac{(4u^3 - 18u^2 + 22u - 6)}{3!}\Delta^4 y_0 + ...]$$

$$f'(51) = \left(\frac{dy}{dx}\right)_{u=0.1} = \frac{1}{10}[16.69 + \frac{(0.2-1)}{2}(5.47) + \left(\frac{(3(0.1)^2) - 6(0.1) + 2}{3!}\right)(-9.23) +$$

$$\left(\frac{(4(0.1)^3 - 18(0.1)^2) + 22(0.1) - 6}{24}\right)(11.99) + ...]$$

$$= \frac{1}{10}\left[16.69 - 2.188 - 2.1998 - 1.9863\right]$$

$$f'(51) = 1.0316$$

$$f''(x) = \left(\frac{d^2y}{dx^2}\right)_{u=0.1} = \frac{1}{h^2}[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{(6u^2 - 18u + 11)}{12}\Delta^4 y_0 + ...]$$

$$f''(51) = \frac{1}{100}\left[5.47 + (0.1-1)(-9.23) + \frac{(6(0.1)^2 - 18(0.1) + 11)}{12}(11.99)\right]$$

$$= \frac{1}{100}\left[5.47 + 8.307 + 9.2523\right]$$

$$f''(51) = 0.2303$$

2. For the given data, find the first two derivative at x=1.1

x	**1.0**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**
y	**7.989**	**8.403**	**8.781**	**9.129**	**9.451**	**9.750**	**10.031**

Solution:

The difference table is as follows

X	y=f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1	7.989						
		0.414					
1.1	8.403		-0.036				
		0.378		0.006			
1.2	8.781		-0.030		-0.002		
		0.348		0.004		0.001	
1.3	9.129		-0.026		-0.001		0.002
		0.322		0.003		0.003	
1.4	9.451		-0.023		0.002		
		0.299		0.005			
1.5	9.750		-0.018		0		

1.6	10.031	0.281					

$$f'(x) = \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_0 + \cdots\right] \quad Where\ u = \frac{x-x_0}{h}$$

$$u = \frac{x-x_0}{h} = \frac{1}{1} = 1$$

$$\left(\frac{dy}{dx}\right)_{x=1.1} = \frac{1}{1}\left[0.414 + \frac{(2(1)-1)}{2!}(-0.036) + \frac{(3(1\,)-6(1)+2)}{3!}(0.006) + \cdots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1.1} = 0.3950$$

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NUMERICAL INTEGRATION BY TRAPEZOIDAL METHOD

3. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by i) Trapezoidal rule ii) Simpson's rule. And compare the result with its actual integration value.

Solution:

Here $y(x) = \frac{1}{1+x^2}$

Let $h = 1$

x :0	1	2	3	4	5	6
y: 1	0.5	0.2	0.1	0.058824	0.038462	0.27027

We know that for Trapezoidal rule

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$\int_0^6 \frac{dx}{1+x^2} = \frac{1}{2}[(1 + 0.27027) + 2(0.5 + 0.2 + 0.1 + 0.058824 + 0.038462)]$$

$$\int_0^6 \frac{dx}{1+x^2} = 1.41079950$$

We know that Simpson's one third rule is

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}[(y_0 + y_5) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$\int_0^6 \frac{dx}{1+x^2} = \frac{1}{3}[(0.5 + 0.027027) + 4(0.5 + 0.1 + 0.038462) + 2(0.2 + 0.58824)]$$

$$\int_0^6 \frac{dx}{1+x^2} = 1.28241$$

We know that Simpson's three – eight rule is

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3}{8}[(1 + 0.027027) + 3(0.5 + 0.2 + 0.058824 + 0.038462) + 2(0.1)]$$

$$\int_0^6 \frac{dx}{1+x^2} = 1.35708188$$

By actual integration,

$$\int_0^6 \frac{dx}{1+x^2} = (tan^{-1}x)_0^6 = tan^{-1}6 = 1.40564764$$

Conclusion:

Here the value by trapezoidal rule is closer to the actual value than the value by Simpson's rule.

4. **Take $h = 0.05$, evaluate $\int_1^{1.3} \sqrt{x}\, dx$ using Trapezoidal rule and Simpson's three-eighth rule.**

Solution:

x	1	1.05	1.1	1.15	1.2	1.25	1.3
y	1	1.0247	1.0488	1.0724	1.0954	1.118	1.1402

We know that for Trapezoidal rule

$$\int_1^{1.3} \sqrt{x}\, dx = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$\int_1^{1.3} \sqrt{x}\, dx = \frac{0.05}{2}[(1 + 1.1402) + 2(1.0247 + 1.0488 + 1.0724 + 1.0954 + 1.118)]$$

$$\int_1^{1.3} \sqrt{x}\, dx = 0.3215$$

We know that Simpson's three – eight rule is

$$\int_{x_0}^{x_0+nh} f(x)\, dx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$\int_{1}^{1.3} \sqrt{x}\, dx = \frac{3(0.05)}{8}[(1 + 1.1402) + 3(1.0247 + 1.0488 + 1.0954 + 1.118) + 2(1.0724)]$$

$$\int_{1}^{1.3} \sqrt{x}\, dx = 0.3215$$

ROMBERG'S METHOD

5. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Romberg's method correct to 4 decimal places. Hence deduce an approximation value of π.

Solution:

Let $y = \frac{1}{1+x^2}$

Let

$$I = \int_{0}^{1} \frac{dx}{1 + x^2}$$

Take h = 0.5. The tabulated values of y are

x:	0	0.5	1
y:	1	0.8	0.5

We know that for Trapezoidal rule

$$\int_{0}^{1} \frac{dx}{1 + x^2} = \frac{h}{2}[(y_0 + y_2) + 2(y_1)]$$

$$\int_{0}^{1} \frac{dx}{1 + x^2} = \frac{0.5}{2}[(1 + 0.5) + 2(0.8)]$$

$$\int_0^1 \frac{dx}{1+x^2} = 0.775 = I_1$$

Take h = 0.25. The tabulated values of y are

x:	0	0.25	0.5	0.75	1
y:	1	0.9412	0.8	0.64	0.5

We know that for Trapezoidal rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0+y_4)+2(y_1+y_2+y_3)]$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{0.25}{2}[(1+0.5)+2(0.9412+0.8+0.64)]$$

$$\int_0^1 \frac{dx}{1+x^2} = 0.7828 = I_2$$

Take h = 0.125. The tabulated values of y are

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.9846	0.9412	0.8767	0.8	0.7191	0.64	0.5664	0.5

We know that for Trapezoidal rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0+y_8)+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)]$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{0.125}{2}[(1+0.5)+2(0.9846+0.9412+0.8767+0.8+0.7191+0.64+0.5664)]$$

$$\int_0^1 \frac{dx}{1+x^2} = 0.78475 = I_3$$

Using Romberg's formula for I_1 and I_2 we have

$$I = I_2 + \left(\frac{I_2 - I_1}{3}\right) = 0.7828 + \left(\frac{0.7828 - 0.775}{3}\right) = 0.7828 + 0.0026$$

$$I = 0.7854$$

Using Romberg's formula for I_2 and I_3 we have

$$I = I_3 + \left(\frac{I_3 - I_2}{3}\right) = 0.78475 + \left(\frac{0.78475 - 0.7828}{3}\right) = 0.78475 + 0.00065$$

I

$$I = 0.7854$$

$$\therefore I = \int_0^1 \frac{dx}{1+x^2} = 0.7854 \qquad \ldots(1)$$

6. Evaluate $\int_0^1 \frac{dx}{1+x}$ correct to three decimal places using Romberg's method. Hence, find the value of log 2.

Solution:

Using Trapezoidal rule, let us find the value of the given definite integral by taking $h = 0.5, 0.25$ and 0.125 respectively,

1. When $h = 0.5$, the values of $y = \frac{1}{1+x}$ are tabulated below.

x:	0	0.5	1
y:	1	0.6666	0.5

We know that for Trapezoidal rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0 + y_2) + 2(y_1)]$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{0.5}{2}[(1 + 0.5) + 2(0.6666)]$$

$$\int_0^1 \frac{dx}{1+x^2} = 0.7083 = I_1$$

ii) Take h = 0.25. The tabulated values of y are

x:	0	0.25	0.5	0.75	1
y:	1	0.8	0.6666	0.5714	0.5

We know that for Trapezoidal rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{0.25}{2}[(1 + 0.5) + 2(0.8 + 0.6666 + 0.5714)]$$

$$\int_0^1 \frac{dx}{1+x^2} = 0.697 = I_2$$

iii) Take h = 0.125. The tabulated values of y are

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.8889	0.8	0.7272	0.6667	0.6153	0.5714	0.5333	0.5

By Trapezoidal rule

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{0.125}{2}[(1 + 0.5) + 2(0.8889 + 0.8 + 0.7272 + 0.6667 + 0.6153 + 0.5714 + 0.5333)]$$

$$\int_0^1 \frac{dx}{1+x^2} = 0.6941 = I_3$$

Using Romberg's formula for I_1 and I_2 we have

$$I = I_2 + \left(\frac{I_2 - I_1}{3}\right) = 0.697 + \left(\frac{0.697 - 0.7083}{3}\right) = 0.6932$$

Using Romberg's formula for I_2 and I_3 we have

$$I = I_3 + \left(\frac{I_3 - I_2}{3}\right) 0.6941 + \left(\frac{0.6941 - 0.697}{3}\right) = 0.6931$$

$$\therefore I = \int_0^1 \frac{dx}{1+x^2} = 0.693 \qquad \ldots (1)$$

Evaluation of $log_e 2$

$$\int_0^1 \frac{dx}{1+x^2} = 0.693$$

i.e., $[\log(1+x)]_0^1 = 0.693 \Rightarrow log_e 2 - log_e 1 = 0.693$

$$log_e 2 = 0.693$$

DOUBLE INTEGRALS USING TRAPEZOIDAL AND SIMPSON'S RULES

7. Evaluate $\int_1^{1.2}\int_1^{1.4} \frac{1}{x+y} dx\, dy$ by using Trapezoidal rule taking h=0.1 and k=0.1

Solution:

y\x	1	1.1	1.2	1.3	1.4
1	0.5000	0.4762	0.4545	0.4348	0.4167
1.1	0.4762	0.4545	0.4348	0.4167	0.4000
1.2	0.4545	0.4348	0.4167	0.4000	0.3846

I = $\frac{hk}{4}$ [(sum of values of f at the four corners)

+ 2 (sum of values of f at the remaining nodes on the boundary)

+ 4(sum of the values of f at the interior nodes)]

$$I = \frac{(0.1)(0.1)}{4}[(0.5000 + 0.4167 + 0.3846 + 0.4545)$$

$$+ 2(0.4762 + 0.4545 + 0.4348 + 0.4000 + 0.4000 + 0.4167 + 0.4348 + 0.4762)$$

$$+ 4(0.4545 + 0.4348 + 0.4167)]$$

$I = 0.0349$

8. **Evaluate $\int_0^1 \int_0^1 \frac{1}{x+y+1} dx\, dy$ by using Trapezoidal rule taking h=0.5 and k=0.25**

Solution:

	0	0.5	1
0	1	0.6667	0.5
0.25	0.8	0.5714	0.4444
0.5	0.6667	0.5	0.40
0.75	0.5714	0.4444	0.3636
1	0.50	.40	0.3333

I = $\frac{hk}{4}$ [(sum of values of f at the four corners)

+ 2 (sum of values of f at the remaining nodes on the boundary)

+ 4(sum of the values of f at the interior nodes)]

I = $\frac{(0.5)(0.25)}{4}$ $[(1 + 0.5 + 0.3333 + 0.5) + 2(0.667 + 0.4444 + 0.40 + 0.3636 + 0.40 + 0.5714 +$

$0.6667 + 0.8) + 4(0.5714 + 0.5 + 0.4444)]$

$= 0.5319$

9. **Evaluate $\int_1^3 \int_1^2 \frac{1}{xy} dx\, dy$ by using Trapezoidal rule taking h=0.5 and k=0.5**

Solution:

	1	1.5	2
1	1	0.667	0.5
1.5	0.667	0.4444	0.3333
2	0.5	0.3333	0.25
2.5	0.4000	0.2667	0.2000
3	0.3333	4.5000	0.1667

I = $\frac{hk}{4}$ [(sum of values of f at the four corners)

+ 2 (sum of values of f at the remaining nodes on the boundary)

+ 4(sum of the values of f at the interior nodes)]

I = $\frac{(0.5)(0.5)}{4}[(1 + 0.5 + 0.3333 + 0.1667) + 2(0.667 + 0.3333 + 0.25 + 0.2 + 4.5 + 0.4 + 0.5 + 0.667) + 4(0.4444 + 0.3333 + 0.2667)]$

$I = 1.3258$

10. Evaluate $\int_0^{1/2}\int_0^{1/2} \frac{sin(xy)}{1+xy}\, dx\, dy$ using Simpson's rule with h=1/4=k

Solution:

Let $f(x,y) = \frac{sin(xy)}{1+xy}$

The values of $f(x,y)$ at the nodal points are given in the following table

	0	1/4	½
0	0	0	0
¼	0	0.0588	0.1108
1/2	0	0.1108	0.1979

By Simpson's rule, I = $\frac{hk}{9}$ [(sum of values of f at the four corners)

+ 2 (sum of the values of f at the odd position on the boundary except the corners)

+ 4 (sum of the values of f at the even position on the boundary)

+ {4 (sum of the values of f at odd positions) + 8 (sum of the values of

f at even positions) on the odd row of the matrix except boundary rows}

+ {8 (sum of the values of f at the odd positions)+16 (sum of the

Values of f at the even position) on the even rows of the matrix}]

$$I = \frac{\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)}{9}[(0+0+0.1979+0)+4(0+0+0.1108+0.1108)+16(0.0588)]$$

$I = 0.0141$

11. Evaluate $\int_1^{1.4}\int_2^{2.4}\frac{1}{xy}\,dx\,dy$ by using Trapezoidal rule taking h=0.1 and k=0.1 and verify with actual integration .

Solution:

y\x	1	1.1	1.2	1.3	1.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3968	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

I = $\frac{hk}{4}$ [(sum of values of f at the four corners)

+ 2 (sum of values of f at the remaining nodes on the boundary)

+ 4(sum of the values of f at the interior nodes)]

$$I = \frac{(0.1)(0.1)}{4}[(0.5000 + 0.4167 + 0.3571 + 0.2976)$$

$$+ 2(0.3846 + 0.4167 + 0.4545 + 0.4762 + 0.4545 + 0.4348 + 0.3788 + 0.3472 + 0.3205$$

$$+ 0.3106 + 0.3247 + 0.3401)$$

$$+ 4(0.4329 + 0.4132 + 0.3953 + 0.3968 + 0.3788 + 0.3623 + 0.3663 + 0.3497 + 0.3344)]$$

$$I = 0.0614$$

By actual integration:

$$\int_1^{1.4} \int_2^{2.4} \frac{1}{xy} dx\, dy = \left(\int_1^{1.4} \frac{1}{y} dy) \, (\int_2^{2.4} \frac{1}{x} dx \right)$$

$$= (log\, y)_1^{1.4} (log\, y)_2^{2.4}$$

$$= (log\ 1.4)[log 2.4 - log 2]$$

$$= log(1.4) log(1.2)$$

$$\int_1^{1.4} \int_2^{2.4} \frac{1}{xy} dx\, dy = 0.0613$$

UNIT-IV

INITIAL VALUE PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS

PART-A

TAYLOR SERIES METHOD

1. **Using Taylor series method fine $y(1.1)$ given that $y' = x + y, y(1) = 0$**

Solution:

Given $y' = x + y$ and $x_0 = 1, y_0 = 0$

We know that Taylor series formula is

$$y_1 = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots$$

$y' = x + y$ $\quad$ $y_0' = 1 + 0 = 1$

$y'' = 1 + y'$ $\quad$ $y_0'' = 1 + 1 = 2$

$y''' = y''$ $\quad$ $y_0''' = 2$

$$y_1 = 0 + (x-1) + \frac{(x-1)^2}{2}(2) + \frac{(x-1)^3}{6}(2)$$

$$y(1.1) = 0 + (1.1-1) + \frac{(1.1-1)^2}{2}(2) + \frac{(1.1-1)^3}{6}(2)$$

$$y_1 = y(1.1) = 0.1103$$

2. **Find $y(0.1)$ if $\frac{dy}{dx} = 1 + y, y(0) = 1$ using Taylor series method.**

Solution:

Given $y' = 1 + y$ and $x_0 = 0, y_0 = 1$

We know that Taylor series formula is

$$y_1 = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots$$

$y' = 1 + y$	$y_0' = 1 + 1 = 2$
$y'' = y'$	$y_0'' = 2$
$y''' = y''$	$y_0''' = 2$

$$y_1 = 1 + (x-0)2 + \frac{(x-0)^2}{2}(2) + \frac{(x-0)^3}{6}(2) + \frac{(x-0)^4}{24}(2)$$

$$= 1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12}$$

$$y(0.1) = 1 + 2(0.1) + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.4)^4}{12}$$

$$y_1 = y(0.1) = 1.2103$$

3. State the advantages and disadvantages of the Taylor's series method.

Solution:

The method gives a straight forward adaptation of classic calculus to develop the solution as an infinite series. It is a powerful single step method if we are able to find the successive derivatives easily.

If f(x,y) involves some complicated algebraic structures then the calculation of higher derivatives becomes tedious and the method fails.

EULER AND MODIFIED EULER METHOD

4. State Euler's method to solve $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

Solution:

$y_1 = y_0 + hf(x_0, y_0)$ where $n = 0,1,2 \ldots$

5. State Modified Euler's method to solve $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.

Solution:

$$y_1 = y_0 + hf\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right)$$

6. Find $y(0.1)$ by using Euler's method given that $\frac{dy}{dx} = x + y, y(0) = 1$.

Solution:

Given $y' = x + y, x_0 = 0, y_0 = 1$

By Euler's method

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = 1 + (0.1)(0 + 1) = 1 + 0.1 = 1.2$$

$$y_1 = y(0.1) = 1.2$$

7. **Find $y(0.2)$ for the equation $y' = y + e^x$, given that $y(0) = 0$ by using Euler's method.**

Solution:

Given $y' = y + e^x, x_0 = 0, y_0 = 0, h = 0.2$

By Euler algorithm, $y_1 = y_0 + hf(x_0, y_0)$

$$= 0 + 0.2f(0,0)$$

$$= 0.2[0 + e^0] = 0.2$$

$$y(0.2) = 0.2$$

RUNGE-KUTTA METHOD FOR SOLVING FIRST AND SECOND ORDER EQUATIONS

8. **State the fourth order Runge-Kutta algorithm.**

Solution:

Let h denote the interval between equidistant values of x. if the initial values are (x_0, y_0), the first increment in y is computed from the formulas.

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$x_1 = x_0 + h, y_1 = y_0 + \Delta y$$

The increment in y in the second interval is computed in a similar manner using the same four formulas, using the values x_1, y_1 in the place of x_0, y_0 respectively.

$$f_1(x, y, z) = z$$

$$f_2(x, y, z) = -xz - y$$

By Runge- Kutta method

$k_1 = hf_1(x_0, y_0, z_0)$

$= (0.1)f_1(0,1,0)$

$= (0.1)(0)$

$k_1 = 0$

$l_1 = hf_2(x_0, y_0, z_0)$

$= (0.1)f_2(0,1,0)$

$= (0.1)(0 - 1)$

$l_1 = -0.1$

MILNE'S PREDICTOR AND CORRECTOR METHODS

9. State Milne's predictor-corrector formula.

Solution:

Milne's Predictor Formula:

$$y_{n+1,\ p} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_{n})$$

Milne's Corrector Formula:

$$y_{n+1,\ c} = y_{n-1} + \frac{h}{3}(2y'_{n-1} - 4y'_{n} + y'_{n+1})$$

10. Distinguish between single step methods and multi-step methods.

Solution:

single step method	multi-step method
Taylor's series, Euler's, Modified Euler's, Runge – Kutta method of fourth order	Milne's and Adams predictor - corrector method
One prior value is required for finding the value of y at x_i	Four prior value are required for finding the value of y at x_i

11. What are multi-step methods? How are they better than single step methods?

Solution:

One step method: We use data of just one proceeding step.

Multi step method: We use data from more than one of the proceeding steps.

PART-B

TAYLOR SERIES METHOD

1. **Find the value of y at $x = 0.1, 0.2$ given that $\frac{dy}{dx} = x^2y - 1, y(0) = 1$,by Tailor's series method up to four terms.**

Solution:

Given $y' = x^2y - 1$ and $x_0 = 0, y_0 = 1$

We know that Taylor series formula is

$$y = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots \qquad \ldots (1)$$

$y' = x^2y - 1$	$y_0' = 0 - 1 = -1$
$y'' = 2xy + x^2y'$	$y_0'' = 2(0)(1) + 0(-1) = 0$
$y''' = 2[xy' + y] + x^2y'' + y'2x$ $= 2y + 4xy' + x^2y''$	$y_0''' = 2(1) + 4(0)(-1) + 0 = 2$
$y^{iv} = 2y' + 4[xy'' + y'] + x^2y''' + y'''2x$ $= 6y' + 6xy'' + x^2y'''$	$y_0^{iv} = 6(-1) + 6(0)(0) + (0)(2)$ $= -6$

Substituting in equation (1) we get

$$y = 1 + (x-0)(-1) + \frac{(x-0)^2}{2}(0) + \frac{(x-0)^3}{6}(2) + \frac{(x-0)^4}{24}(-6)$$

$$y = 1 - x + \frac{x^3}{3} - \frac{x^4}{4}$$

To find y (0.1)

$$y(0.1) = 1 - 0.1 + \frac{0.1^3}{3} - \frac{0.1^4}{4}$$

$$y(0.1) = 1 - 0.1 + 0.00033 - 0.000025$$

$$y(0.1) = 0.900305$$

To find y (0.2)

$$y(0.2) = 1 - 0.2 + \frac{0.2^3}{3} - \frac{0.2^4}{4}$$

$$y(0.2) = 1 - 0.2 + 0.0026 + -0.0004$$

$$y(0.2) = 0.8022$$

$$x_0 = 0.1\ , y_0 = 00.0993, h = 0.1$$

$$y_2 = y(0.2) = 0.09933 + (0.1)(0.9801334) + \frac{(0.1)^2}{2}(-0.3946868) + \frac{(0.1)^3}{6}(-3.84159)$$

$$y(0.2) = 0.19467$$

2. Determine the value of $y(0.4)$ using milnes's method given $y' = xy + y^2$,$y(0) = 1$. Using Taylor series method obtain the values of $y(0.1)$ and y(0.2) and y(0.3).

Solution :

Given $y' = xy + y^2$ and $x_0 = 0, y_0 = 1,$

By Taylor series formula is

$$\boldsymbol{y = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots} \qquad \ldots (1)$$

$y' = \ xy + y^2$	$y_0' = 1$
$y'' = xy' + y + 2\,y'$	$y_0'' = 1 + 2(1)(1) = 1$
$y''' = xy' + y' + y' + 2yy'' + 2y'y'$ $= xy' + 2y'^2 + 2yy'' + 2y'$	$y_0''' = 2 + 6 + 2 = 10$
$y^{iv} = xy''' + y'' + 2y'' + 2y'y'' + 2y\,y''' + 4\ y'y''$ $= xy''' + 3y'' + 4\ y'y'' + 2y\,y'''$	$y_0^{iv} = 9 + 12 + 20 = 41$

Substituting in equation (1) we get

$$y_1 = y(0.1) = 1 + 0.1(1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(10) + \frac{(0.1)^4}{24}(41)$$
$$y(0.1) = 1.11684$$

$$y_2 = y(0.2) = 1 + 0.2(1) + \frac{(0.2)^2}{2}(3) + \frac{(0.2)^3}{6}(10) + \frac{(0.2)^4}{24}(41)$$
$$y(0.2) = 1.276067$$

$$y_3 = y(0.3) = 1 + 0.3(1) + \frac{(0.3)^2}{2}(3) + \frac{(0.3)^3}{6}(10) + \frac{(0.)^4}{24}(41)$$
$$y(0.1) = 1.48384$$

X	0	0.1	0.2	0.3
Y	1	1.11684	1.27607	1.49384

$$y_{4,\ p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$y_{4,\ p} = 1 + \frac{4(0.1)}{3}[2(1.35902) - 1.88357 + 2(2.67974)]$$

$$y_{4,\ p} = 1.82586$$

$$y_4' = (0.4)1.82586 + 1.82586^2 \quad = 4.06411$$

By Mile's corrector formula is

$$y_{4,\ c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 1.27607 + \frac{0.1}{3}[1.88357 + (2.67974) + 4.06411]$$

$$y_{4,\ c} = 1.83096$$

$$y_4 = 1.83096$$

3. Using Taylor series method fin y at x=1.1 by solving the equation if $\frac{dy}{dx} = x^2 + y^2, y(1) = 2$. Carryout the computations upto fourth order derivative.

Solution:

Given initial condition $x_0 = 1, y_0 = 2, h = 0.1$

We know that Taylor series formula is

$$y = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots \qquad \ldots (1)$$

$y' = x^2 + y^2$	$y_0' = 1 + 2 = 3$
$y'' = 2x + 2yy'$	$y_0'' = 2(1) + 2(2)(3) = 14$
$y''' = 2 + 2yy" + 2y'^2$	$y_0''' = 2 + 2(2)(14) + 2(3)^2 = 76$
$y^{iv} = 2\ yy''' + 2y'y" + 4y'y"$	$y_0^{iv} = 2(2)(76) + 2(3)(14) + 4(3)(14) = 556$

Substituting in equation (1) we get

$$y_1 = 2 + \frac{(x-x_0)}{1!}(1) + \frac{(x-x_0)^2}{2!}(2) + \frac{(x-x_0)^3}{3!}(8) + \frac{(x-x_0)^4}{4!}(28) + \cdots$$

$$y_1 = 2 + \frac{(1.1-1)}{1!}(1) + \frac{(1.1-1)^2}{2!}(2) + \frac{(1.1-1)^3}{3!}(8) + \frac{(1.1-1)^4}{4!}(28) + \cdots$$

$$y(1.1) = 2 + 0.1(3) + \frac{(0.1)^2}{2}(14) + \frac{(0.1)^3}{6}(76) + \frac{(0.1)^4}{24}(556) + \cdots$$

$$y_1 = 2 + 0.3 + 0.07 + 0.0127 + 0.00232 = 2.3850$$

EULER AND MODIFIED EULER METHOD

4. Apply Modified Euler's method to find $y(0.2)$ and $y(0.4)$ given that $\frac{dy}{dx} = x^2 + y^2, y(0) = 1$ by taking h=0.2

Solution:

Initial conditions are

$$x_0 = 0, y_0 = 1, h = 0.2$$

By Euler algorithm

$$y_{n+1} = y_n + hf(x_n + \frac{h}{2}, y_n + \frac{h}{2}(x_n, y_n')$$

$Let\ n = 0$

$$y_1 = y_0 + hf(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}(x_0, y_0')$$

$$= 1 + (0.2)f(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}(\mathbf{0^2} + \mathbf{1^2}))$$

$$= 1 + (0.2)f(0.1,1.1)$$

$$= 1 + (0.2)[(\mathbf{0.1})^2 + (\mathbf{1.1})^2]$$

$$= 1 + (0.2)(1.22)$$

$$= 1.244$$

$$y_1 = 1.244$$

$$\boldsymbol{y_1 = y(0.2) = 1.244}$$

$Let\ n = 1,$

$$x_1 = 0.2, y_1 = 1.244, h = 0.2$$

$$y_2 = y_1 + hf(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}(x_{1}, y_1')$$

$$= 1.244 + (0.2)f(0.2 + \frac{0.2}{2}, 1.244 + \frac{0.2}{2}((\mathbf{0.2})^2 + (\mathbf{1.244})^2))$$

$$= 1.005 + (0.2)f(0.3,1.4028)$$

$$= 1.005 + (0.2)[(\mathbf{0.3})^2 + (\mathbf{1.3684})^2]$$

$$y_2 = 1.6365$$

$$y_2 = y(0.4) = 1.6365$$

5. Evaluate y at $x = 0.2$ given $\frac{dy}{dx} = y - x^2 + 1,\ y(0) = 0.5$ using modified Euler's method.

Solution:

$$\frac{dy}{dx} = y - x^2 + 1, \qquad x_0 = 0, \qquad y_0 = 0.5\ , h = 0.2$$

By Euler algorithm

$$y_{n+1} = y_n + hf(x_n + \frac{h}{2}, y_n + \frac{1}{2}h(x_n, y_n'))$$

$Let n = 0$

$$y_1 = y_0 + hf(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}h(x_0, y_0'))$$

$$= 0.5 + (0.2)f\left(0 + \frac{0.2}{2}, 0.5 + \frac{0.2}{2}(0,0.5)\right)$$

$f(x_0, y_0) = y_0 - x_0^2 + 1, \mathrm{f}(0,0.5) = 0.5 + 0 + 1 = 1.5$

$$= 0.5 + (0.2)f[(0.1, 0.5 + 0.1(1.5)]$$

$$= 0.5 + (0.2)f(0.1,0.65)$$

$$f(0.1,0.65) = 0.65 + (0.1)^2 + 1 = 0.65 - 0.01 + 1$$

$$= 1.65 - 0.01 = 1.64$$

$$y_1 = 0.5 + (0.2)(1.64)$$

$$0.5 + 0.328 = 0.828$$

$$\boldsymbol{y(0.2) = 0.828}$$

6. Apply Modified Euler's method to find $y(0.1)$ and $y(0.2)$ given that $\frac{dy}{dx} = x^2 + y^2, y(0) = 1$

Solution:

Initial conditions are

$$x_0 = 0, y_0 = 1, h = 0.1$$

By Euler algorithm

$$y_{n+1} = y_n + hf(x_n + \frac{h}{2}, y_n + \frac{1}{2}h(x_n, y_n'))$$

$Let\ n = 0$

$$y_1 = y_0 + hf(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}h(x_0, y_0'))$$

$$= 1 + (0.1)f(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}(\mathbf{1} + \mathbf{0}))$$

$$= 1 + (0.1)f(0.05, 1.05)$$

$$= 1 + (0.1)\left[(\mathbf{0.05})^{\mathbf{2}} + (\mathbf{1.05})^{\mathbf{2}}\right]$$

$$= 1 + (0.1)(1.105)$$

$$= 1.1105$$

$$y_1 = 1.1105$$

$$\boldsymbol{y_1 = y(0.1) = 1.1105}$$

$Let\ n = 1,$

$$x_1 = 0.1, y_1 = 1.1105, h = 0.1$$

$$y_2 = y_1 + hf(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}h(x_1, y_1'))$$

$$= 1.1105 + (0.1)f(0.1 + \frac{0.1}{2}, 1.1105 + \frac{0.1}{2}((\mathbf{0.2})^{\mathbf{2}} + (\mathbf{1.1105})^{\mathbf{2}}))$$

$$= 1.1105 + (0.1)f(0.15, 1.27321)$$

$$= 1.1105 + 0.1((\mathbf{0.15})^{\mathbf{2}} + (\mathbf{1.27321})^{\mathbf{2}}))$$

$$y_2 = 1.2749$$

$$\boldsymbol{y_2 = y(0.2) = 1.2749}$$

RUNGE-KUTTA METHOD FOR SOLVING FIRST AND SECOND ORDER EQUATIONS

7. Use Runge-Kutta method of order 4 to find y(1.1) given $\frac{dy}{dx} = \boldsymbol{y^2 + xy}$, $\boldsymbol{y(1) = 1}$,

Solution:

Given $\frac{dy}{dx} = \boldsymbol{y^2 + xy}$, $x_0 = 1, y_0 = 1$and $h = 0.1$

By Runge-kutta method

$$\boldsymbol{k_1 = hf(x_0, y_0)}$$

$$= (0.1)f(1,1)$$

$$= (0.1)(11 + 1)$$

$$k_1 = 0.2$$

$$\boldsymbol{k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)}$$

$$= (0.1)f\left(1 + \frac{0.1}{2}, 1 + \frac{0.2}{2}\right)$$

$$= (0.1)f(1.05,1.1)$$

$$k_2 = 0.2365$$

$$\boldsymbol{k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)}$$

$$= (0.1)f\left(1 + \frac{0.1}{2}, 1 + \frac{0.2365}{2}\right)$$

$$= (0.1)f(1.05,1.118)$$

$$k_3 = 0.2423$$

$$\boldsymbol{k_4 = hf(x_0 + h, y_0 + k_3)}$$

$$= (0.1)f(1 + 0.1,1 + 0.2423)$$

$$= (0.1)f(1.1,1.12423)$$

$$k_4 = 0.2909$$

$$\Delta \boldsymbol{y} = \boldsymbol{\frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$= \frac{1}{6}(0.2 + 2(0.0.2365) + 2(0.2423) + 0.2909)$$

$$\Delta y = 0.2414$$

$$y_1 = y_0 + \Delta y$$

$$= 1 + 0.2414$$

$$\boldsymbol{y(1.05) = 1.2414}$$

To find y(1.1):

Here $x_1 = 1.05, y_1 = 1.2414$and $h = 0.1$

$$\boldsymbol{k_1 = hf(x_1, y_1)}$$

$$= (0.1)f(1.05,1.2414)$$

$$= (0.1)(2.84454)$$

$$k_1 = 0.28445$$

$$\boldsymbol{k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)}$$

$$= (0.1)f\left(1.05 + \frac{0.1}{2}, 1.2414 + \frac{0.28445}{2}\right)$$

$$= (0.1)f(1.1,1.3836)$$

$$k_2 = 0.27133$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= (0.1)f\left(1.05 + \frac{0.1}{2}, 1.2414 + \frac{0.27133}{2}\right)$$

$$= (0.1)f(1.1,1.37706)$$

$$k_3 = 0.34110$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= (0.1)f(1.05 + 0.1,1.2421 + 0.34110)$$

$$= (0.1)f(1.15,1.5825)$$

$$k_4 = 0.43241$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.2844 + 2(0.27133) + 2(0.34110) + 0.43241)$$

$$\Delta y = 0.3236016$$

$$y_2 = y_1 + \Delta y$$

$$= 1.2414 + 0.3236016$$

$$y(1.1) = 1.565001$$

MILNE'S PREDICTOR AND CORRECTOR METHODS

8. **Use Milne's predictor – corrector formula to find $y(0.4)$**

Given $\frac{dy}{dx} = \frac{(1+x^2)y^2}{2}, y(0) = 1, y(0.1) = 1.06, y(0.2) = 1.12 \; and \; y(0.3) = 1.21$

Solution:

Given $\frac{dy}{dx} = y' = \frac{1}{2}(1 + x^2)y^2 \; and \; h = 0.1$

$$x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, x_4 = 0.4, x_5 = 0.5$$

$$y_0 = 1, y_1 = 1.06, y_2 = 1.12, y_3 = 1.21, y_4 = ?$$

Milene's Predictor formula we have,

$$y_{n+1}, p = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n \dots \dots \dots \dots \dots \dots \dots (1)$$

To get y_4, put $n = 3$ in (1) we get

$$y_4, p = y_0 + \frac{4h}{3}[2y_1' - y'_2 + 2y'_3] \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots (2)$$

$$y'_1 = \left[\frac{1}{2}(1 + x^2)y^2\right]$$

$$= \frac{1}{2}(1 + {x_1}^2){y_1}^2$$

$$= \frac{1}{2}[1 + (0.1)^2](1.06)^2$$

$$y'_1 = 0.56742\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ .(3)$$

$$y'_2 = \frac{1}{2}(1 + {x_2}^2){y_2}^2$$

$$= \frac{1}{2}[1 + (0.2)^2](1.12)^2$$

$$= \frac{1}{2}(1 + 0.04)(1.2544)$$

$$y'_2 = 0.6529 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ (4)$$

$$y'_3 = \frac{1}{2}(1 + {x_3}^2){y_3}^2$$

$$= \frac{1}{2}[1 + (0.3)^2](1.21)^2$$

$$= \frac{1}{2}[1 + 0.09](1.464)$$

$$y'_3 = 0.79793 \ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ (5)$$

Substituting (3),(4) and (5) in (2) we get,

$$y_{4,p} = 1 + \frac{4(0.1)}{2}[2(0.56742) - 0.65229 + 2(0.79793)]$$

$$= 1 + \frac{0.4}{3}[1.13484 - 0.65229 + 1.56586]$$

$$= 1 + 0.27712$$

$$\boldsymbol{y(0.4) = 1.27712}$$

Milne’s corrector formula we have

$$\boldsymbol{y_{n+1,c} = y_{n-1} + \frac{h}{3}\left(y'_{n-1} + 4y'_n + y_{n+1}\right)}$$

To get y_4, put $n = 3$ we get

$$y_{4,c} = y_2 + \frac{h}{3}(y'_2 + 4y_3 + y'_4) \ldots \ldots \ldots \ldots\ldots \ldots \ldots \ldots (6)$$

$$y'_4 = \frac{1}{2}(1 + {x_4}^2){y_4}^2$$

$$= \frac{1}{2}[1 + (0.4)^2](1.27712)^2$$

$$= \frac{1}{2}(1 + 0.16)(1.63104)$$

$$= \frac{1}{2}(1.16)(1.63104)$$

$$= 0.94600 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (7)$$

Substituting (4), (5), (7) in (6) we get,

$$y_{4,c} = 1.12 + \frac{0.1}{3}[0.65229 + 4(0.79793) + 0.94600]$$

$$= 1.12 + \frac{0.1}{3}[4.79001]$$

$$= 1.12 + 0.159667$$

$$\boldsymbol{y(0.4) = 1.27966}$$

9. Using Milne's predictor and corrector formulae , find $y(4.4)$ given

$$\boldsymbol{5xy' + y^2 - 2 = 0, y(4) = 1, y(4.1) = 1.0049, y(4.2) = 1.0097, y(4.3) = 1.0143}$$

Solution:

Given $y' = \frac{2-y^2}{5x}, x_0 = 4, x_1 = 4.1, x_2 = 4.2, x_3 = 4.3, x_4 = 4.4$

$$y_0 = 1, y_1 = 1.0049, y_2 = 1.0097, y_3 = 1.0143$$

$$y'_1 = \frac{2 - y_1^2}{5x_1} = \frac{2 - (1.0049)^2}{5(4.1)} = 0.0493$$

$$y'_2 = \frac{2 - y_2^2}{5x_2} = \frac{2 - (1.0097)^2}{5(4.2)} = 0.0467$$

$$y'_3 = \frac{2 - y_3^2}{5x_3} = \frac{2 - (1.0143)^2}{5(4.3)} = 0.0452$$

By Mile's predictor formula is

$$y_{4,\ p} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

$$y_{4,\ p} = 1 + \frac{4(0.1)}{3}[2(0.0493 - 0.0467 + 2(0.0452)]$$

$$y_{4,\ p} = 1.01897$$

$$y_4' = \frac{2 - y_4^2}{5x_4} = \frac{2 - (1.1897)^2}{5(4.4)} = 0.0437$$

By Mile's corrector formula is

$$y_{4,\ c} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$y_{4,\ c} = 1.0097 + \frac{0.1}{3}[0.0467 + 4(0.0452 + 0.0437]$$

$$y_{4,\ c} = 1.01874$$

$$= 1 + \frac{4(0.1)}{3}[2(1.3552) - 1.8535 + 2(2.6589)]$$

$$y_{4,\ p} = 1.8233$$

$$y_4' = x_4y_4 + {y_4}^2 = (0.4)(1.8233) + (1.8233)^2 = 4.0537$$

By Mile's corrector formula is

$$y_{4,\ c} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$y_{4,\ c} = 1.2774 + \frac{0.1}{3}[1.8535 + 4(2.6589) + 4.0537]$$

$$y_{4,\ c} = 1.8165$$

10. Using Runge-kutta method of fourth order, find y for $x = 0.1, 0.2, 0.3$ given that $\frac{dy}{dx} = xy + y^2, y(0) = 1$ Continue the solution at x=0.4 using Milne's method .

Solution:

Given $\frac{dy}{dx} = xy + y^2, x_0 = 0, y_0 = 1, h = 0.1$

By Runge –kutta method

$$k_1 = hf(x_0, y_0)$$

$$= (0.1)f(0,1)$$

$$= (0.1)(\mathbf{0} + \mathbf{1})$$

$$k_1 = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= (0.1)f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$$

$$= (0.1)f(0.05, 1.05)$$

$$= (0.1)\big((0.05)(1.05) + (1.05)^2\big)$$

$$k_2 = 0.1155$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= (0.1)f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1155}{2}\right)$$
$$= (0.1)f(0.05, 1.50775)$$
$$= (0.1)\big((0.05)(1.50775) + (1.50775)^2\big)$$
$$k_3 = 0.1172$$
$$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$= (0.1)f(0 + 0.1, 1 + 0.1172)$$
$$= (0.1)f(0.1, 1.4424)$$
$$= (0.1)\big((0.1)(1.4424) + (1.4424)^2\big)$$
$$k_4 = 0.1260$$
$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}(0.1 + 2(0.1155) + 2(0.1172) + 0.1260)$$
$$\Delta y = 0.1152$$
$$y_1 = y_0 + \Delta y$$
$$= 1 + 0.1152$$
$$\boldsymbol{y(0.1) = 1.1152}$$

To find y(0.2):

Here $x_1 = 0.1, y_1 = 1.1152$

$$k_1 = hf(x_1, y_1)$$
$$= (0.1)f(0.1, 1.1152)$$
$$= (0.1)\big((0.1)(1.1152) + (1.1152)^2\big)$$
$$k_1 = 0.1255$$
$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= (0.1)f\left(0.1 + \frac{0.1}{2}, 1.1152 + \frac{0.1255}{2}\right)$$
$$= (0.1)f(0.05, 1.1780)$$
$$= (0.1)\big((0.05)(1.1780) + (1.1780)^2\big)$$
$$k_2 = 0.1355$$
$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= (0.1)f\left(0.1 + \frac{0.1}{2}, 1 + \frac{0.1355}{2}\right)$$
$$= (0.2)f(0.05, 1.1355)$$
$$= (0.1)\big((0.05)(1.1355) + (1.1355)^2\big)$$
$$k_3 = 0.1577$$
$$k_4 = hf(x_1 + h, y_1 + k_3)$$
$$= (0.1)f(0.1 + 0.1, 1.1152 + 0.1577)$$
$$= (0.1)f(0.2, 1.2729)$$
$$= (0.1)\big((0.2)(1.2729) + (1.2729)^2\big)$$
$$k_4 = 0.1875$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}(0.1255 + 2(0.1355) + 2(0.1577) + 0.1875)$$
$$\Delta y = 0.1499$$
$$y_2 = y_1 + \Delta y$$
$$= 11152 + 0.1499$$
$$\boldsymbol{y(0.2) = 1.2651}$$

To find y(0.3):

Here $x_2 = 0.2, y_2 = 1.2651$

$$k_1 = hf(x_2, y_2)$$
$$= (0.1)f(0.2,1.2651)$$
$$= (0.1)\left((0.2)(1.2651) + (1.2651)^2\right)$$
$$k_1 = 0.1853$$
$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right)$$
$$= (0.1)f\left(0.2 + \frac{0.1}{2}, 1.2651 + \frac{0.1}{2}\right)$$
$$= (0.1)f(0.25,1.3578)$$
$$= (0.1)\left((0.25)(1.3578) + (1.3578)^2\right)$$
$$k_2 = 0.2183$$
$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)$$
$$= (0.1)f\left(0.2 + \frac{0.1}{2}, 1.2651 + \frac{0.2183}{2}\right)$$
$$= (0.1)f(0.25,1.3742)$$
$$= (0.1)\left((0.25)(1.3742) + (1.3742)^2\right)$$
$$k_3 = 0.2232$$
$$k_4 = hf(x_2 + h, y_2 + k_3)$$
$$= (0.1)f(0.2 + 0.1,1.2651 + 0.2232)$$
$$= (0.1)f(0.3,1.4883)$$
$$= (0.1)\left((0.1)(1.4883) + (1.4883)^2\right)$$
$$k_4 = 0.2662$$
$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}(0.1853 + 2(0.2183) + 2(0.2232) + 0.2662)$$
$$\Delta y = 0.2224$$
$$y_3 = y_2 + \Delta y$$
$$= 1.2651 + 0.2224$$
$$y(0.3) = 1.4875$$
$$x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, x_4 = 0.4$$
$$y_0 = 1, y_1 = 1.1152, y_2 = 1.2651, y_3 = 1.4875, y_4 = ?$$
$$y' = xy + y^2$$

$$y_0' = x_0y_0 + {y_0}^{\mathbf{2}} = (0)(1) + (1)^{\mathbf{2}} = \mathbf{1}$$
$$y_1' = x_1y_1 + {y_1}^{\mathbf{2}} = (0.1)(1.1152) + (1.1152)^{\mathbf{2}} = 1.3552$$
$$y_2' = x_2y_2 + {y_2}^{\mathbf{2}} = (0.2)(1.2651) + (1.2651)^{\mathbf{2}} = 1.8535$$
$$y_3' = x_3y_3 + {y_3}^{\mathbf{2}} = (0.3)(1.4875) + (1.4875)^{\mathbf{2}} = 2.6589$$

By Mile's predictor formula is

$$\boldsymbol{y_{4,\ p} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']}$$
$$y_{4,\ p} = 1 + \frac{4(0.1)}{3}[2(1.3552) - 1.8535 + 2(2.6589)]$$
$$y_{4,\ p} = 1.8233$$
$$y_4' = x_4y_4 + {y_4}^{\mathbf{2}} = (0.4)(1.8233) + (1.8233)^{\mathbf{2}} = 4.0537$$

By Mile's corrector formula is

$$\boldsymbol{y_{4,\ c} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']}$$
$$y_{4,\ c} = 1.2651 + \frac{0.1}{3}[1.8535 + 4(2.6589) + 4.0537]$$
$$y_{4,\ c} = 1.8165$$

UNIT-V

BOUNDARY VALUE PROBLEMS IN ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

PART – A

CLASSIFICATION OF PDE OF SECOND ORDER

1. **Classify the following equation :** $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0.$

Solution:

Given $u_{xx} + 4u_{xy} + 4u_{yy} - u_x + 2u_y = 0$

Here A = 1, B = 4, C = 4

Condition is $B^2 - 4AC = 16 - 4(1)(4) = 0$

The given equation is parabolic.

2. **Classify the partial differential equation** $u_{xx} + 2u_{xy} + 4u_{yy} = 0, x, y > 0$

Solution:

Given $u_{xx} + 2u_{xy} + u_{yy} = 0$

Here A = 1, B = 2, C = 4

Condition is $B^2 - 4AC = 4 - 4(1)(4) = -12 < 0$

The given equation is elliptic

3. **Classify the pde** $u_{xx} - xu_{yy} = 0.$

Solution:

Given $u_{xx} - xu_{yy} = 0$

Here A=1, B=0, C=$-x$

Condition is $B^2 - 4AC = 0 - 4(1)(-x) = 4x = +ve$

The given equation is Hyperbolic if x>0,

Elliptic if x<0,

Parabolic if x=0.

NUMERICAL SOLUTION OF ODE BY FINITE DIFFERENCE METHOD

4. **What is the central difference approximation for y" and y'**

Solution:

$$y_i^{'} = \frac{y_{i+1} - y_{i-1}}{2h}$$

$$y_i" = \frac{y_{i-1} - 2y_i + y_i + 1}{h^2}$$

where i=1, 2, 3….n

and $nh = b - a$(ie., upper limit – lower limit)

5. **Obtain the finite difference scheme for the differential equation $2y" + y = 5$**

Solution:

$$y"(x) - \frac{1}{2}y(x) = \frac{5}{2}$$

$$y" = \frac{y_{i-1} - 2y_i + y_i + 1}{h^2}$$

$$\frac{y_{i-1} - 2y_i + y_i + 1}{h^2} - \frac{1}{2}y_i = \frac{5}{2}$$

$$y_{i-1} - 2y_i + y_i + 1 - \frac{1}{2}h^2 y_i = \frac{5}{2}h^2$$

$$y_{i-1} - \left[2 + \frac{1}{2}h^2\right] y_i + y_i + 1 = \frac{5}{2}h^2$$

$$2y_{i-1} - [4 + h^2]y_i + 2y_i + 1 = 5h^2$$

ONE DIMENSIONAL HEAT EQUATION BY EXPLICIT AND IMPLICIT METHODS

6. **Write down the Crank – Nicholson formula to solve parabolic equation (OR)**

State Crank-Nicholson's difference scheme.

Solution:

$$\frac{1}{2}\lambda u_{i+1,j+1} + \frac{1}{2}\lambda u_{i-1,j+1} - (\lambda+1)u_{i,j+1} = -\frac{1}{2}\lambda u_{i+1,j} - \frac{1}{2}\lambda u_{i-1,j} + (\lambda-1)u_{i,j}$$

$$(or)$$

$$\lambda\left(u_{i+1,j+1} + u_{i-1,j+1}\right) - 2(\lambda+1)u_{i,j+1} = 2(\lambda-1)u_{i,j} - \lambda\left(u_{i+1,j} + u_{i-1,j}\right)$$

7. Write down Bender-Schmidt's difference scheme in general form and using suitable value of λ, (or) Give the Bender-Schmidt recurrence equation (or) Give the explicit finite difference scheme for $\frac{\partial^2 u}{\partial x^2} = a\frac{\partial u}{\partial t}$.

Solution:

$$u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}$$

if $= \frac{1}{2}$,

$$u_{i,j+1} = \frac{1}{2}[u_{i-1,j} + u_{i+1,j}]$$

TWO DIMENSIONAL LAPLACE EQUATIONS

8. Write down the standard five point formula to find the numerical solution of Laplace equation.

Solution:

The Standard five point formula: [SFPF]

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$

9. Write down the diagonal five point formula to solve the Laplace's equation $\nabla^2 u(x,y) = 0$.

Solution :

The diagonal five-point formula is,

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1}\right]$$

10. What is the error for solving Laplace's and Poisson's equations by finite difference method?

Solution:

The error in replacing $\frac{\partial^2 u}{\partial x^2}$ by the difference expression is of the order $o(h^2)$. Since $h = k$,

the error in replacing $\frac{\partial^2 u}{\partial y^2}$ by the difference expression is of the order (h^2)

<u>TWO DIMENSIONAL POISSON EQUATIONS</u>

11. Write the difference "scheme for solving the Poisson equation $\nabla^2 u = f(x,y)$

Solution:

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh)$$

PART - B

FINITE DIFFERENCE SOLUTION OF SECOND ORDER ORDINARY EQUATION

1. Solve $y'' - y = 0$ with boundary conditions y(0)=0 and y(1)=1 taking $h = 0.25$

Solution:

Divide the interval [0,1] into four equal sub intervals

$x_0 = 0,$

$x_1 = 0.25,$

$x_2 = 0.5,$

$x_3 = 0.75,$

$x_4 = 1$

The finite-difference approximation of the given equation is

$$y_{i-1} - 2y_i + y_{i+1} = h^2 y_i$$

$$16y_{i-1} - 33y_i + 16y_{i+1} = 0 \ldots\ldots\ldots\ldots\ldots..(1)$$

$$y_0{}' = 0 \Rightarrow \frac{y_1 - y_{-1}}{2h} = 0 \Rightarrow y_1 = y_{-1}$$

For i = 0 $\quad (1) \Rightarrow 16y_{-1} - 33y_0 + 16y_1 = 0 \;\Rightarrow -33y_0 + 32y_1 = 0 \ldots\ldots\ldots\ldots\ldots(2)$

For i=1 $\quad (1) \Rightarrow 16y_0 - 33y_1 + 16y_2 = 0 \ldots\ldots\ldots\ldots\ldots(3)$

For i=2 $\quad (1) \Rightarrow 16y_1 - 33y_2 + 16y_3 = 0 \ldots\ldots\ldots\ldots\ldots(4)$

For i=3 $\quad (1) \Rightarrow 16y_2 - 33y_3 = -16 \ldots\ldots\ldots\ldots\ldots(5)$

From (2), (3), (4) and (5) we get $-y_0 + 0.97y_1 = 0 \ldots\ldots\ldots\ldots\ldots(6)$

$$y_0 - 2.062y_1 + y_2 = 0 \ldots\ldots\ldots\ldots\ldots(7)$$

$$y_1 - 2.062y_2 + y_3 = 0 \ldots\ldots\ldots\ldots\ldots (8)$$

$$y_2 - 2.062y_3 = -1 \ldots\ldots\ldots\ldots\ldots (9)$$

$(6) + (7) \Rightarrow \quad -1.092y_1 + y_2 = 0 \ldots\ldots\ldots\ldots\ldots (10)$

$8 * 1.092 \Rightarrow \quad 1.092y_1 - 2.252y_2 + 1.092y_3 = 0 \ldots\ldots\ldots\ldots\ldots (11)$

$(10) + (11) \Rightarrow \quad -1.252y_2 + 1.092y_3 = 0 \ldots\ldots\ldots\ldots\ldots (12)$

$(12) + (9) * 1.252 \Rightarrow \quad -1.49y_3 = -1.252$

$$y_3 = 0.84$$

$$(9) \quad \Rightarrow y_2 = 0.732$$

$$(8) \quad \Rightarrow y_1 = 0.67$$

$$(6) \quad \Rightarrow y_0 = 0.65$$

Hence $y(0) = 0.65,$

$y(0.25) = 0.67,$

$y(0.5) = 0.732,$

$y(0.75) = 0.84,$

$y(1) = 1$

FINITE DIFFERENCE SOLUTION OF ONE DIMENSIONAL HEAT EQUATION BY EXPLICIT AND IMPLICIT METHODS

1. **Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, subject to $u(0,t) = u(1,t) = 0$ and $u(x,0) = \sin \pi x, 0 < x < 1$ and $h = 0.2$ using Bender-schmidt method. Find the value of u up to $t = 0.1$.**

Solution:

Since h and k are not given Bender – Schmidt method.

$$k = \frac{a}{2}h^2 = \frac{h^2}{2} \qquad \because a = 1$$

Since range of x is (0, 1), take h = 0.2

Hence $k = \frac{(0.2)^2}{2} = 0.02$

The formula is $u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right)$

We form the table

j \ i	0	0.2	0.4	0.6	0.8	1
0	0	0.5878	0.9511	0.9511	0.5878	0
0.02	0	0.4756	0.7695	0.7695	0.4756	0
0.04	0	0.3848	0.6225	0.6225	0.3848	0
0.06	0	0.3113	0.5036	0.5036	0.3113	0
0.08	0	0.2518	0.4074	0.4074	0.2518	0
0.1	0	0.2037	0.3296	0.3296	0.2037	0

2. Solve by Crank Nicolson's method $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ **for 0<x<1, t>0 given that** $u(0, t) =$

$\mathbf{0}, \boldsymbol{u}(\mathbf{1}, \boldsymbol{t}) = \mathbf{0}, \boldsymbol{u}(\boldsymbol{x}, \mathbf{0}) = \mathbf{100}\boldsymbol{x}\,(\mathbf{1} - \boldsymbol{x})$**.Compute u for one time step with h=1/4 and k=1/64 .**

Solution:

From the given equation

$$u_{xx} = u_t$$

$$a = 1$$

$$h = 1/4 = 0.25$$

$$k = 1/64$$

t/x	0	0.25	0.5	0.75	1
0	0	18.75	25	18.75	0
1/64	0	u_1	u_2	u_3	0

$$u_1 = \frac{1}{4}[0 + 25 + 0 + u_2] \Rightarrow 4u_1 - u_2 = 25 \ldots\ldots(1)$$

$$u_2 = \frac{1}{4}[18.75 + 18.75 + u_1 + u_3] \Rightarrow -u_1 + 4u_2 - u_3 = 37.5 \ldots(2)$$

$$u_3 = \frac{1}{4}[25 + 0 + 0 + u_2] \Rightarrow -u_2 + 4u_3 = 25 \ldots\ldots(3)$$

By solving (1), (2) & (3)

$$u_1 = 9.8214$$

$$u_2 = 14.2857$$

$$u_3 = 9.8214$$

3. **Solve $\boldsymbol{u_t} = \boldsymbol{u_{xx}}$ in $\mathbf{0} < x < 5, t > 0$ given that $\boldsymbol{u}(\mathbf{0}, \boldsymbol{t}) = \mathbf{0}, \boldsymbol{u}(\mathbf{5}, \boldsymbol{t}) = \mathbf{0}, \boldsymbol{u}(\boldsymbol{x}, \mathbf{0}) = \boldsymbol{x}^2(\mathbf{25} - \boldsymbol{x}^2)$. Compute u upto t=2 with $\Delta x = \mathbf{1}$ by using Bender-Schemidth formula.**

   Solution:

$$u_{xx} = au_t \ldots\ldots\ldots\ldots(1)$$

$$u_{xx} = u_t \ldots\ldots \quad \ldots\ldots(2)$$

Form (1)and (2)

$$a = 1$$

Given $a = 1, h = 1$

$$k = \frac{1}{2}(1)^2 = \frac{1}{2} = 0.5$$

The formula is $u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right)$

We form the table

j/i	0	1	2	3	4	5
0	0	24	84	144	144	0
0.5	0	42	84	114	72	0
1	0	42	78	78	57	0
1.5	0	39	60	67.5	39	0
2	0	30	53.25	49.5	33.75	0
2.5	0	26.625	39.75	43.5	24.75	0
3	0	19.875	35.0625	32.25	21.75	0

4. Obtain the Crank-Nicholson finite difference method by taking $\boldsymbol{\lambda = \frac{kc^2}{h^2} = 1}$.

Hence, find u(x,t) in the rod for two times steps for the heat equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, given $\boldsymbol{u(x,0) = \sin(\pi x), u(0,t) = 0, u(1,t) = 0. Take\ h = 0.2}$.

Solution:

From the given equation a=1

$$x \to 0\ to\ 1$$

H=0.2

$$k = ah^2 = (1)(0.2)^2 = 0.04$$

We use

$$u_{i,j+1} = \frac{1}{4}\left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j} + u_{i-1,j}\right]$$

t\x	0	0.2	0..4	0.6	0.8	1
0	0	0.59	0.95	0.95	0.59	0
0.04	0	u_1	u_2	u_3	u_4	0
0.08	0	u_5	u_6	u_7	u_8	0

$$u_1 = \frac{1}{4}[0 + 0 + 0.95 + u_2] \Rightarrow 4u_1 = 0.95 + u_2 \ldots\ldots..(1)$$

$$u_2 = \frac{1}{4}[0.59 + 0.95 + u_1 + u_3] \Rightarrow 4u_2 = 1.54 + u_1 + u_3 \ldots.(2)$$

$$u_3 = \frac{1}{4}[0.95 + 0.59 + u_2 + u_4] \Rightarrow 4u_3 = 1.54 + u_2 + u_4 \ldots\ldots.(3)$$

$$u_4 = \frac{1}{4}[0 + 0 + 0.95 + u_3] \Rightarrow 4u_4 = 0.95 + u_3 \ldots\ldots\ldots\ldots\ldots.(4)$$

Solving (1), (2), (3) & (4)

$$u_1 = 0.3228, u_2 = 0.3411$$

$$u_3 = -0.4984, u_4 = 0.1129$$

$$u_5 = \frac{1}{4}[0 + 0 + u_6 + u_2] \Rightarrow 4u_5 = 0.3411 + u_6$$

$$4u_5 - u_6 = 0.3411 \ldots\ldots\ldots\ldots (5)$$

$$u_6 = \frac{1}{4}[u_5 + u_7 + u_1 + u_3] \Rightarrow 4u_6 = -0.1756 + u_5 + u_7$$

$$4u_6 + u_5 + u_7 = -0.1756 \ldots. (6)$$

$$u_7 = \frac{1}{4}[u_6 + u_8 + u_2 + u_4] \Rightarrow 4u_7 = 0.454 - u_6 - u_8$$

$$4u_7 - u_6 - u_8 = 0.454 \ldots\ldots. (7)$$

$$u_8 = \frac{1}{4}[u_7 + 0 + 0 + u_3] \Rightarrow 4u_8 = -0.4984 + u_7$$

$$4u_8 + u_7 = -0.4984 \ldots\ldots\ldots\ldots\ldots. (8)$$

Solving (5), (6), (7) & (8)

$u_5 = 0.0854$, $u_6 = 0.00062$, $u_7 = 0.0927$, $u_8 = -0.1014$

<u>ONE DIMENSIONAL WAVE EQUATION</u>

5. Solve the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, 0x < 1, t > 0$ satisfying the conditions $u(x,0) = 0, \frac{\partial u}{\partial t}(x,0) = 0$, $u(0,t) = 0$ and $u(1,t) = \frac{1}{2}\sin \pi t$. Compute $u(x,t)$ for 4 time-steps by taking $h = \frac{1}{4}$.

Solution:

Given $u(x,0) = 0$,

$$\frac{\partial u}{\partial t}(x,0) = 0,$$

$$u(0,t) = 0$$

and $u(1,t) = \frac{1}{2} \sin \pi t$

j/i	0	0.25	0.5	0.75	1
0	0	0	0	0	0
0.25	0	0	0	0	0.3537
0.5	0	0	0	0.3537	0.5
0.75	0	0	0.3537	0.5	0.3532
1	0	0.3537	0.5	0.3532	0

TWO DIMENSIONAL POISSON EQUATIONS

6. Solve $\nabla^2 u = 8x^2y^2$ in the square region $-2 \le x, y \le 2$ with u=0 on the boundaries after dividing the region into 16 sub-intervals of length 1 unit.

Solution:

Here $h = 1$. The region of solution of the given Laplace's equation with the boundary values are given in the table

0 0 0 0 0

	u_1	u_2	u_3
	u_4	u_5	u_6
	u_7	u_8	u_9

0 0 0 0 0

Let $u_1, u_{2,} u_3 \dots\dots u_9$ be the values of u at the interior grid points.

The Poisson P.D.E $\nabla^2 u = 8x^2y^2$ is symmetrical about x and y axes and also about the line $y = x$.

Hence we have $u_1 = u_3 = u_7 = u_9$ and $u_2 = u_4 = u_6 = u_8$.

Hence we have to find u_1, u_2, u_5 only.

The standard five point formula is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} - 4u_{i,j} = 8i^2j^2 \dots\dots\dots\dots\dots\dots (1)$$

Using (1) at $u_7 (i = -1, j = -1)$ we have

$$u_{-2,-1} + u_{0,-1} + u_{-1,0} - 4u_{-1} - 1 = 8(-1)^2(-1)^2$$

$$0 + u_8 + 0 + u_4 - 4u_7 = 8$$

$$u_2 + u_2 - 4u_1 = 8$$

$$u_2 - 2u_1 = 4 \dots\dots\dots\dots\dots\dots (2)$$

Using (1) at u_2 $(i = 0, j = 1$ we have

$$u_{-1,1} + u_{1,1} + u_{0,0} + u_{0,2} - 4u_{0,1} = 8(0)(1)$$

$$u_1 + u_3 + u_5 + 0 - 4u_2 = 0$$

$$2u_1 - 4u_2 + u_5 = 0\ldots \ldots \ldots .. .. \ldots (3)$$

Using (1) at $u_5(i = 0, j = 0)$ we have

$$u_{-1,0} + u_{1,0} + u_{0,-1} + u_{0,1} - 4u_{0,0} = 0$$

$$u_4 + u_6 + u_8 + u_2 - 4u_5 = 0$$

$$4u_2 - 4u_5 = 0$$

$$u_2 = u_5\ldots \ldots \ldots .. .. \ldots (4)$$

Solving these equations (2),(3),(4) we get

$$u_1 = -3, u_2 = -2, u_5 = -2$$

∴ the solution to the given Poisson equation at the 9 interior mesh points are

$$u_1 = u_3 = u_7 = u_9 = -3$$

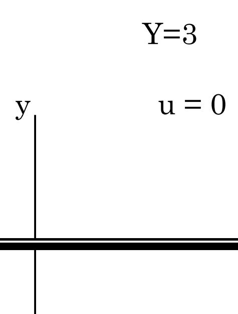
$$u_2 = u_4 = u_6 = u_8 = -2 \text{ and } u_5 = -2$$

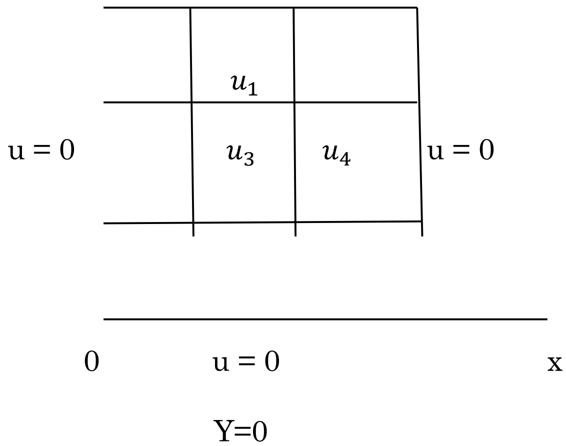
7. Solve the Poisson equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square mesh with sides $x = 0, y = 0, x = 3$ *and* $y = 3$ *with* $u = 0$ on the boundary and mesh length 1 unit.

(OR)

Solve $\nabla^2 u = -10(x^2 + y^2 + 10)$ in the square region $0 \le x, y \le 3$ *with* $u = 0$ on the boundary and mesh length 1 unit.

Solution:





Let the value of u at the four mesh points A, B, C and D be u_1, u_2, u_3, u_4 respectively. The differential equation is

$$\nabla^2 u = -10(x^2 + y^2 + 10) \qquad \ldots(1)$$

Replacing $\nabla^2 u$ by the finite difference expressions and putting x = ih, y = ih (h = 1) in (1), we get

$$u_{i-1,,j} - 2u_{i,j} + u_{i+1,j-1} - u_{i,j-1} - 2u_{i',j} + u_{i,j+1} = -10(i^2 + j^2 + 10) \qquad \ldots(2)$$

Applying the formula (1) at A [where I = 1, j =2]

$$0 + 0 + u_2 + u_3 - 4u_1 = -10(1 + 4 + 10)$$

$$u_2 + u_3 - 4u_1 = -150 \qquad \ldots(3)$$

Applying the formula (1) at B where i = 2, j = 2

$$u_1 + 0 + 0 + u_4 - 4u_2 = -10(4 + 4 + 10)$$

$$u_1 + u_4 - 4u_2 = -180 \qquad \ldots(4)$$

Applying the formula (1) at C where i = 1, j = 1

$$0 + u_1 + u_4 + 0 - 4u_3 = -10(1 + 1 + 10)$$

$$u_1 + u_4 - 4u_3 = -120 \qquad \ldots (5)$$

Applying the formula (1) at C where i = 1, j = 1

$$u_3 + u_2 + 0 + 0 - 4u_4 = -10(4 + 1 + 10)$$

$$u_3 + u_2 - 4u_4 = -150 \qquad \ldots (6)$$

$$u_1 = \frac{1}{4}[u_2 + u_3 + 150] \qquad \ldots (7)$$

$$u_2 = \frac{1}{4}[u_1 + u_4 + 180] \qquad \ldots (8)$$

$$u_3 = \frac{1}{4}[u_1 + u_4 + 120] \qquad \ldots (9)$$

$$u_4 = \frac{1}{4}[u_2 + u_3 + 150] \qquad \ldots (10)$$

From (7) and (10)

By using Gauss Seidel method, we can solve the above equation.

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